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P-solutions for a Type of Structured Interval Parametric Systems

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Abstract. *It is known that many analysis problems arising in science and engineering, which contain parametric uncertainties, can be modeled by a system of linear algebraic equations where the elements of the matrix and right-hand side vector depend linearly on a number of interval parameters. In certain cases, the linear interval parametric (LIP) system has a specific structure. Recently, an efficient iterative method for a type of such structured LIP systems has been suggested which yields the approximate solution in the form of an interval vector. In the present paper, it is shown that the latter method can be modified in an appropriate manner to produce new so-called p-solutions of the LIP system considered. The properties of the p-solutions are analyzed. It is shown that the p-solutions can be used to construct new methods for determining various enclosing solutions of structured systems.*

1. Introduction

As is known, various problems encountered in science and engineering involve uncertain parameters that are given as intervals. A large class of such problems can be formulated in (or reduced to) the form of a (real square) linear interval parameter (LIP) system of size n

(1a) $A(p)x=b(p),\ p \in \mathbf{p}$

whose elements $a_{ij}(p)$ and $b_i(p)$ are affine linear functions

(1b) $a_{ij}(p)=\alpha_{ij}+\sum_{(\mu=1)}^{m} a_{ij\mu}p_\mu$, $b_i(p)=\beta_i+\sum_{(\mu=1)}^{m}=1\beta_{i\mu}p_\mu$, $p \in \mathbf{p_i}$

where $p=(p_1,..,p_m)$ is a real m-dimensional vector belonging to a given interval vector $\mathbf{p}=(\mathbf{p_1},..,\mathbf{p}_m)$. Let

$S^x=\{x:A(p)x=b(p),\ p \in \mathbf{p}\}$

denote the solution set of (1). As is well known, the following "solutions" to (1) are considered: (*i*) outer solution $\boldsymbol{x}$: an interval vector containing S^x, (*ii*) hull solution $\boldsymbol{x}^*$: the narrowest $\boldsymbol{x}$ enclosing S^x, and (*iii*) inner estimation of the hull (IEH) solution x^{in} : an interval vector such that $x^{in} \subseteq x^*$.

Various methods for determining the above solutions are known [1-9].

Sometimes the problem to be solved is composed of the LIP system (1) and an additional relationship (e.g., [9])

(2) $z=f(x, p)$

where z is a n_1-dimensional vector of "secondary" variables; the function f can be, in general, nonlinear in x and p. Now let

$S^z=\{z: z=f(x,p),\ A(p)x=b(p),\ p \in \mathbf{p}\}$

be the solution set of the pair (1), (2). On the basis of S^z, all three types of solutions $\boldsymbol{z}$, $\boldsymbol{z}^*$ and $\boldsymbol{z}^{in}$ can be defined. In this more general setting, the solutions related to S^x are called primary interval solutions whereas those related to S^z are referred to as secondary interval solutions.

It is important to underline that all interval solutions are interval vectors assessing from outward or inward the respective solution sets S^x or S^z.

Meanwhile it was observed that in some cases the LIP system considered has a specific structure. Thus, in mechanical engineering if finite element method (FEM) is used for analysis of truss structures, system (1a) has the form [10]

(3) $(K+BD(p)A)x=a+Fq,\ p \in \mathbf{p},\ q \in \mathbf{q}$

where the interval parameters p appear only in the diagonal matrix D while the additional independent parameters q are solely in the right-hand side. A substantial generalization has been considered in [11] where the standard system (1a) is written (by renaiming the parameters that appear in the right-hand side of the system) as

(4a) $A(p)x=b(p, q),\ p \in \boldsymbol{p},\ q \in \boldsymbol{q}$,

(4b) $A(p)=A_0+\sum_{(\mu=1)}^{m_1} A_\mu p_\mu$, $b(p,q)=b_0+\sum_{(\mu=1)}^{m_1}=b_\mu p_\mu+\sum_{(\mu=m_1+1)}^{m} b_\mu p_\mu$.

An efficient iterative method for finding an outer interval solution to (4) has been proposed in [11] which yields an interval vector $\mathbf{x}$ enclosing the solution set S^x of (4).

Recently, a new type of solution to (1), so called parameterized or p-solution, has been introduced in [12]. It is defined as a corresponding linear interval form

(5) $\mathbf{x}(p)=Lp+\mathbf{a},\ p \in \mathbf{p}$

where L is a real $n \times m$ matrix while $\boldsymbol{a}$ is an n-dimensional interval vector. It should be stressed that the parameter vector $\boldsymbol{p}$ in (5) is a symmetric vector of unit radius, i.e., $\mathbf{p}_j=[-1,1]$ for $j=1,...,m.$. Iterative methods for determining $\boldsymbol{x}(p)$ were suggested in [12-14]; a direct method for determining a p-solution was proposed in [15]. (For the advantages of $\boldsymbol{x}(p)$ over the standard (non-parametric) solutions $\boldsymbol{x}$, the reader is referred to [12] – [16]).

Remark 1. For simplicity the same notation $\boldsymbol{p}$ is used in formulae (1) and (5) for different intervals (centered or not). Relevant changes in the notations will be made later in Section 2.2.

In the present paper, we show that the method of [11] can be modified to produce a p-solution of (4). That is achieved by replacing, at each iteration, the involved interval arithmetic (IA) operations with corresponding affine arithmetic (AA) operations (e.g., [17]) (Section 2). Next (in Section 3), the properties of the p-solutions are presented. It is shown that the p-solutions can be used to construct new methods for determining narrower primary $\boldsymbol{x}$ or secondary $\boldsymbol{z}$ solutions. A numerical example illustrating the superiority of the new approach over other known methods is given in Section 4.

1. Determining the *p*-solutions

2.1. The iterative method of [11]

It is based on Theorems 3 and 4 of [11]. The former theorem shows under what conditions and how a LIP system of the general form (1a) can be transformed into an equivalent form

(6) $(A_0+LD(p)R)x=b_0+LD(p)t+Fq$

where L, R and F are matrices of sizes $n\times k_1$, $k_1\times n$ and $n\times(K-k_1)$ respectively, whereas t is a vector of size K. Explicit formulae for computing L, R, F and t are given in [11].

The interval method proper of [11] employs interval arithmetic (therefore referred to as method MI) and is based on Theorem 4 in [4]. Let $p \in \boldsymbol{p}$, $q \in \boldsymbol{q}$ and D_0 be the centre of $D(p)$. Assume that $C=(A_0+LD_0\ R)^{-1}$ exists. Also let w be a unit vector (all elements are equal to one). Now, compute

(7a) $w'=w-|D_0-\mathbf{D}||RCL|w,$

(7b) $w''=|D_0-\mathbf{D}||RCb_0+RCF\mathbf{q}+RCLD_0\ t-t|.$

If w'>0, then methods MI is applicable and has the following computational scheme:

Step 1. Using (7), compute the initial value of the interval vector

(8) $\mathbf{d}=[-\alpha w, \alpha w]$, $\alpha=\max(w_i''/w_i')$.

Step 2. Iterate

(9a) $\mathbf{y}=\{RCb_0+(RCF)\mathbf{q}+(RCL)(D_0\ t+\mathbf{d})\}\cap\mathbf{y}$,

(9b) $\mathbf{d}=\{(D_0-\mathbf{D})(\mathbf{y}-t)\}\cap\mathbf{d}$.

until some stopping criterion is satisfied.

Step 3. Compute the interval vector

(10) $\mathbf{x}=Cb_0+(CF)\mathbf{q}+(CL)(D_0t+\mathbf{d})$

which (as shown in [11]) encloses the solution set S^x of (4).

The stopping criterion used in method MI is the same as that of [10]: the iterations are terminated either as soon as the sum of the components of $\boldsymbol{d}$ for two consecutive iterations does not improve by a factor of 0.999 or after at most 10 iterations.

The method MI, whenever applicable, seems to be the best known method for computing an interval outer solution of (1) for the following reasons:

– it does not require strong regularity of A(p) on $\boldsymbol{p}$;

– the method is capable of solving problems for large parameter intervals;

– it is more general and more efficient than the method of [10].

2.2. The Method for Computing a *p*-solution

We suggest a method for enclosing the solution set S^x of (1) (or S^z of (1), (2)) when system (1a) can be represented in the form (6). The method (referred to as method MA) reduces, essentially, to replacing the interval arithmetic (IA) operations in method MI by affine arithmetic (AA) operations.

In accordance with Remark 1 we first denote the interval parameter vectors in (4) as $\boldsymbol{p'}$ and $\boldsymbol{q'}$. Next, we define the corresponding affine forms $\langle p'\rangle$ and $\langle q'\rangle$. We also introduce the centered affine forms

$\langle p\rangle:=-(\langle p'\rangle-\check{p})$, $\langle q\rangle:=\langle q'\rangle-\check{q}$.

Accounting for the specificity of AA, we introduce the following modifications in the computational scheme of methods MI:

M1. Remove the intersections in (9).

M2. Replace the IA operations in (9a) by their AA counterparts to obtain $\langle y'\rangle$. More specifically, the auxiliary affine form $\langle q_a'\rangle=(RCF)\langle q'\rangle$ is first introduced. Next, the centre is found to obtain.

$\langle y\rangle=\check{y}+\langle q_a\rangle+(RCL)\langle d\rangle$

M3. Replace the multiplication of the diagonal matrix $(D_0-\boldsymbol{D})$ and the vector $(\boldsymbol{y}-t)$ by the Hadamard product of $\langle p\rangle$ and $(\langle y\rangle+t')$, $t'=\check{y}-t$ (the Hadamard product a.*b of two vectors a and b yields a vector c with components $c_i=a_i*b_i$).

Thus, the algorithm of method MA is as follows:

Step 1. Introduce the interval vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and the corresponding affine forms.

Step 2. Introduce $\boldsymbol{d}$ and the corresponding affine form $\langle d\rangle$.

Step 3. Iterate

(11a) $\langle y\rangle=\check{y}+\langle q_a\rangle+(RCL)\langle d\rangle$

(11b) $\langle d\rangle=\langle p\rangle.*(\langle y\rangle+t')$

until a convergence criterion for $\langle d\rangle$ is activated.

Step 4. Compute the affine vector

(12) $\langle x\rangle=Cb_0+(CF)\langle q'\rangle+(CL)(D_0t+\langle d\rangle)$.

Each component of $\langle x\rangle$ is of the form

(13) $\langle x_i\rangle=\check{x}_i+\sum_{j=1}^{m_1}e_{ij}\xi_j+\sum_{j=m_1+1}^{m}e_{ij}\xi_j+\sum_{j=m+1}^{m_t}e_{ij}\xi_j$

where $\xi_j=[-1,1]$, $\check{x}_i$ is the centre of $\langle x_i\rangle$ while e_{ij} are the so-called error terms and mt denotes the total number of the error terms [17].

If we solve a primary problem, $\langle x\rangle$ is written in LIP form as

(14) $\mathbf{x}(p,q)=\check{x}+L^p\ p+L^q\ q+[-\hat{x},\hat{x}]$, $p\in\mathbf{p}$, $q\in\mathbf{q}$

where the $n\times m_1$ and $n\times(m-m_1)$ matrices L^p and L^q are associated (row-wise) with the first two sums in (13), $\boldsymbol{p}$ and $\boldsymbol{q}$ are symmetric intervals of unit radius and $\hat{x}$ is computed component-wise as the radius of the last sum in (13). The form (14) is the sought p-solution of (4).

If the problem considered is of the secondary type (1), (2)

(15) $z=f(x,p,q)$,

then we proceed as follows. We use (13) and AA in (15) to compute the affine vector $\langle z\rangle$ with components

(16) $\langle z_i\rangle=\check{z}_i+\sum_{(j=1)}^{m_1} e_{ij}{}'\xi_j+\sum_{j=k_1+1)}^{m} e_{ij}{}'\xi_j+\sum_{(j=K+1)}^{m_t} e_{ij}{}'\xi_j$.

Next $\langle z\rangle$ is written in the associated LIP form

(17) $z(p,q)=\check{z}+L^p p+L^q q+[-\hat{z},\hat{z}]$, $p\in\mathbf{p}$, $q\in\mathbf{q}$

(the elements of L^p and L^q are now related to the corresponding terms in (16)). Obviously, (17) determines the *p*-solution related to the secondary problem (4), (15).

2. Properties of the *p*-solutions

3.1. Primary Variables

Let *u* denote the augmented parameter vector (p,q) of size m. The properties of (14) follow from Lemmas 1, 2 and Theorem 1 in [12]. First, the range ***x(u)*** of ***x(u)*** over ***u*** provides an outer solution ***x*** of (4), i.e. ***x=x(u)***. Hence the ith component $\boldsymbol{x}_i^*$ of the IH solution x^* to (1) is contained in the ith component of the range, i.e.

(18) $\mathbf{x}_i^*\subset\mathbf{x}_i(\mathbf{u})$.

The inclusions (18) can be improved bounding the ends of the hull solution. Following [12] we define

(19a) $e_i^{(l)}=[\underline{e}_k^l,\ \bar{e}_k^l]$, $\underline{e}_k^l=\underline{x}_k$, $\bar{e}_k^l=\check{x}_k-\sum_j|l_{kj}|$; or $\bar{e}_k^l=\underline{x}_k+\hat{x}_k$;

in a similar way

(19b) $e_i^{(u)}=[\underline{e}_k^u,\ \bar{e}_k^u]$, $\underline{e}_k^u=\check{x}_k+\sum_j|l_{kj}|$ or $\bar{e}_k^l=\overline{x}_k-\hat{x}_k$, $\bar{e}_k^u=\overline{x}_k$.

Theorem 1. *[12]. Let* $\mathbf{e}_i^{(l)}$ *and* $\mathbf{e}_i^{(u)}$ *be the intervals defined by (19; also let* $\underline{x}_i^*$ *and* $\overline{x}_i^*$ *be the endpoints of* x_i^**. Then*

(20) $\underline{x}_i^*\in e_i^{(l)}$, $\overline{x}_i^*\in e_i^{(u)}$.

Here we suggest better enclosures $\boldsymbol{e}_i^{(l)}$ and $\boldsymbol{e}_i^{(u)}$ than those in (19). Beforehand some preliminary facts are needed. Geometrically, the interval vector $\boldsymbol{u}'$ is a box in R^m. A vertex $u^{(v)}$ is a particular combination of the ends of the intervals $\boldsymbol{u}_j'$, *j=l,...,m*. Recall [9] that determining a component $x_k^*=[\underline{x}_k^*,\overline{x}_k^*]$ of x^* is carried out by solving the following pair of global optimization problems:

(21a) $\underline{x}_k^*=\min e_k^T x$,

(21b) $\overline{x}_k^*=\max e_k^T x$,

subject to the constraint (4) (e_k is the *k*th column of the identity matrix). According to Corollary 3 in [11], $\underline{x}_k^*$ and $\overline{x}_k^*$ are attained at corresponding vertices $(u')^l$ and $(u')^u$. As in [9] such problems will be said to have the vertex property. Let

(22) $x_k(u)=\check{x}_k+\sum_{j=1}^{m} l_{kj}u_j+[-\hat{x}_k,\hat{x}_k]$, $u_j\in[-1,1]$

be the *k*th component of the *p*-solution of (4). At this point we make the following assumption.

Assumption 1. It is assumed that all $l_{kj}\neq 0$.

We now introduce two sign vectors s_k^l and s_k^u whose components are

(23a) $s_{kj}^l=-\text{sign}(l_{kj})$, $s_{kj}^u=\text{sign}(l_{kj})$,

as well as two vertices $(u')^l$ and $(u')^u$ with components

(23b) $(u_j')^l=\check{u}_j+s_{kj}^l\hat{u}_j$, $(u_j')^u=\check{u}_j+s_{kj}^u\hat{u}_j$.

Next, the IEH (inner) assessments $\underline{x}_k^{in}$ and $\overline{x}_k^{in}$ are found as the solution of the respective systems

(23c) $A((p')^l)x=b((u')^l)$, $A((p')^u)x=b((u')^u)$,

where $(p')^l$ and $(p')^u$ are the respective partition parts of $(u')^l$ and $(u')^u$. On account of (23), tight bounds $e_k^l=[\underline{e}_k^l,\bar{e}_k^l]$ and $e_k^u=[\underline{e}_k^u,\bar{e}_k^u]$ on $\underline{x}_k^*$ and $\overline{x}_k^*$, respectively, can be computed with

(23d) $\underline{e}_k^l=\underline{x}_k$, $\bar{e}_k^l=\overline{x}_k^{in}$, $\bar{e}_k^l=\overline{x}_k^{in}$, $\bar{e}_k^u=\overline{x}_k$.

Theorem 2. *Under Assumption 1, let* $\boldsymbol{e}_i^{(l)}$ *and* $\boldsymbol{e}_i^{(u)}$ *be the intervals defined by (23). Then*

(24) $\underline{x}_k^*\in\mathbf{e}_k^{(l)}$, $\overline{x}_k^*\in\mathbf{e}_k^{(u)}$.

The above intervals are narrower than the intervals (19), (20).

We now show that tight bounds of the type (23) can be determined even if Assumption 1 is not satisfied. Indeed, let I_1 denote the set of those indices for which $l_{kj}\neq 0$ while I_2 is the set of the remaining indices with $l_{kj}=0$. Also, introduce two real vectors $\underline{u}$ and $\bar{u}$ whose components are guaranteed to take on end-point values according to (23b) for those indices with $j\in I_1$. Thus, each vector u' can be partitioned into two parts as follows:

(25a) $u'=(\underline{u}^{(1)},\ (u')^{(2)})$

or

(25b) $u'=(\bar{u}^{(1)},\ (u')^{(2)})$

where only the components $(u')_j^{(2)}$ of $(u')^{(2)}$, $j\in I_2$ are allowed to vary within the corresponding intervals $(\mathbf{u}_j')^{(2)}$. Let I_1 and I_2 have m_1 and m_2 members, respectively. It is seen that in view of (25) the original interval vector $\mathbf{u}'$ has been reduced to a new m_2 – dimensional interval vector $(\mathbf{u}')^{(2)}$.

We first consider partition (25a) and present a procedure for determining $\underline{x}_k^{in}$.

Procedure 1. Set up the modified system

(26a) $A(\underline{p}^{(1)},\ (p')^{(2)})x=b(\underline{u}^{(1)},\ (u')^{(2)})$

and find its *p*-solution with components

(26b) $\mathbf{x}_k^l\,(u^{(2)})=\check{x}_k^l+\sum_{j\in I_2} l_{kj}{}'u_j+[-\hat{x}_k^l,\ \hat{x}_k^l]$.

If all $l_{kj}^l\neq 0$, the corresponding components $(u_j^{(2)})^l$ forming the solution $\underline{u}^{(2)}$ are determined using (23b) for $j\in I_2$. Thus, we have found the *m*-dimensional vector $\underline{u}'=(\underline{u}^{(1)},\ \underline{u}^{(2)})$. Finally, $\underline{x}_k^{in}$ is determined as the solution of

(26c) $A(\underline{p}')x=b(\underline{u}')$.

If in (26b) some $l_{kj}^l=0$, then $(u')^{(2)}$ is treated as a new parameter vector u' of size m_2. Now u' is partitioned again into two parts according to (25a) and a new attempt to eventually obtain the final $\underline{x}_k^{in}$ is made.

Procedure 2 (for determining $\overline{x}_k^*$). It has, essentially, the same structure as Procedure 1. We now set up the modified system

(27a) $A(\overline{p}^{(1)},\ (p')^{(2)})x=b(\bar{u}^{(1)},\ (u')^{(2)})$

and find its *p*-solution with components

(27b) $\mathbf{x}_k^u(u^{(2)})=\check{x}_k^u+\sum_{j=1}^{m_2} l_{kj}^u u_j+[-\hat{x}_k^u, \hat{x}_k^u]$.

If all $l_{kj}^u \neq 0$, then the correct values $\bar{u}^{(2)}$ are fixed using (23a), (23b). The final parameter vector $\bar{u}'=(\bar{u}^{(1)}, \bar{u}^{(2)})$ yielding $\overline{x}_k^{in}$ is found as the solution of

(27c) $A(\vec{p}')x=b(\bar{u}')$.

If in (27b) some $l_{kj}^u=0$, then $(u')^{(2)}$ is again split into two parts and a new attempt to attain the final $\overline{x}_k^{in}$ is made.

3.2. Secondary Variables

We only consider the problem of bounding the ends of the hull solution z_k^* for the secondary variable

(28a) $z_k=f_k(x,u)$, $u\in\mathbf{u}$.

In the case of a secondary problem, the vertex property of $\underline{z}_k^*$ and $\bar{z}_k^*$ cannot be guaranteed. Nevertheless, tight bounds $\mathbf{e}_k^l=[\underline{e}_k^l, \bar{e}_k^l]$ and $\mathbf{e}_k^u=[\underline{e}_k^u, \bar{e}_k^u]$ on $\underline{z}_k^*$ and $\bar{z}_k^*$, respectively, can be computed. Thus, using the *k*th component of the *p*-solution

(28) $z_k(u)=\check{z}_k+\sum_{j=1}^{m} l_{kj}u_j+[-\hat{z}_k, \hat{z}_k]$, $u\in\mathbf{u}$,

and the interval vector $z_k=[\underline{z}_k, \bar{z}_k]$ to which (28) reduces, we define

(29a) $\underline{e}_k^l=\underline{z}_k$, $\bar{e}_k^l=\check{x}_k-\sum_j |l_{kj}|$;

in a similar way

(29b) $\underline{e}_k^u=\check{x}_k+\sum_j |l_{kj}|$, $\bar{e}_k^u=\bar{z}_k$.

Better results for $\bar{e}_k^l$ and $\underline{e}_k^u$ can be obtained if the signs s_{kj}^l and s_{kj}^u related to l_{kj} in (28) are used. In this case, using the vertex $(u')^l$ with components $(u_j')^l=\check{u}_j+s_{kj}^l\hat{u}_j$, we first compute x_k^l as the solution of $A((p')^l)x=b((u')^l)$. Then we find

(30a) $\bar{e}_k^l=f(x_k^l, (u')^l)$.

The bound $\underline{e}_k^u$ is obtained in a similar way (using the vertex $(u')^u$ and the solution x_k^u of $A((p')^u)x=b((u')^u)$) as

(30b) $\underline{e}_k^u=f(x_k^u, (u')^u)$.

Remark 2. It should be underlined that the bounds (26) and (30) are narrower than those given in [12] and can be used for unstructured systems as well.

Remark 3. It should be stressed that the bounds (23d) and (29) are used for determining the hull solutions x^* or z_k^* in conjunction with constraint propagation techniques [12] – [15]. Thus, reducing the bounds width will enhance the efficiency of the methods for computing the respective hull solutions.

3. Example

We consider a class C_{req} of linear interval parameter DC circuits made up of resistors r_μ and (ideal) voltage sources e_μ, $\mu=1,..,m_1$ as well as current q_μ sources, $\mu=m_1+1,...,m$ where r_μ and q_μ belong to respective intervals $\boldsymbol{r}_\mu'$ and $\boldsymbol{q}_\mu'$ while the voltage sources e_μ are known exactly (are constans). We address the problem of tolerance analysis of such circuits using nodal analysis (NA) or modified nodal analysis (MNA) equations. On introduction of interval conductances $\mathbf{p}_\mu'=1/\mathbf{r}_\mu'$, the corresponding LIP system is of the form

(31a) $G(p)x=I(p)+q$, $p\in\mathbf{p}'$, $q\in\mathbf{q}'$.

where the elements $G_{ii}(p)$ and $G_{ij}(p)=G_{ji}(p)$ of $G(p)$ are the proper and mutual conductance while $I_i(p)$ is the corresponding equivalent current source; x is the vector of the node voltages. System (31a) can be written in the equivalent form

(31b) $G(p)=\sum_{\mu=1}^{m} A^{(\mu)}p_\mu$, $I(p,q)=\sum_{\mu=1}^{m_1} b^{(\mu)}p_\mu+\sum_{\mu=m_1+1}^{m} b^{(\mu)}q_\mu$.

It can be seen that now, in view of (31a), each $A^{(\mu)}$ is a dyad

(31c) $A^{(\mu)}=\xi_\mu\xi_\mu^T$

where ξ_μ is an *n*-dimensional vector. If p_μ is across two nodes *k* and *l*, then ξ_μ is a zero (column) vector except for the entries $\xi_{\mu k}=-1$ and $\xi_{\mu l}=1$; if *l* is the datum node the only nonzero entry is $\xi_{\mu k}=1$. Thus, $A^{(\mu)}$ is a rank-one matrix. Also, it can be shown that $b^{(\mu)}$ have at most two non-zero entries. It is seen that the LIP system considered has a highly specific structure. It turns out that the following result is valid.

Theorem 3. *Let the linear interval parameter DC circuits considered are of class C_{req}. Then the describing system (31) can be transformed into system (4) where L is given by the incidence matrix of the circuit, $R=L^T$ and the components t_μ of t are equal to e_μ or $-e_\mu$ depending on whether the directions of source and current in the μth branch are the same or not.*

The validity of Theorem 3 follows directly from Theorem 3 in [11].

We take up the DC circuit considered in [18], [11] to illustrate the new results obtained in §§ 2.2 and 3. The algorithms of the methods used were programmed in MATLAB environment using the toolbox IntLab.8 [18] to carry out the interval calculations involved. The AA arithmetic was implemented by the *affari* toolbox. The program was run on a 1.3 GHz double-core PC computer.

The DC circuit has m=11 branches and n=5 nodes (not including the datum node). Every resistor has a nominal resistance $\check{r}_\mu=100\Omega$ and an equal tolerance radius $\hat{r}_\mu$ which is defined as a certain percentage of $\check{r}_\mu$, i.e.

(32) $\hat{r}=\hat{r}_\mu=\tau\check{r}_\mu$, $\mu=1,...,m$.

We have fixed $\tau=0.1$. The source voltages are $e_1=e_2=100V$, $e_5=e_7=10V$. We note that in this example the parameter vector q is not present. The LIP system for the circuit considered (obtained by NA) is

(33a) $G(p)x=I(p)$, $p\in\mathbf{p}'$,

(33b) $\mathbf{p}_\mu'=1/\mathbf{r}_\mu'$, $\mathbf{r}_\mu'=\check{r}_\mu+[-\hat{r}_\mu, \hat{r}_\mu]$, $\mu=1,...,m$.

$$G_{11}(p)=p_1+p_3+p_6;\ G_{12}(p)=-p_3;\ G_{15}(p)=-p_6;$$
$$G_{21}(p)=G_{12}(p);\ G_{22}(p)=p_2+p_3+p_4;\ G_{23}(p)=-p_4-p_5;$$
(33c) $$G_{32}(p)=G_{23}(p);\ G_{33}(p)=p_4+p_5+p_7+p_{10};\ G_{34}(p)=-p_7;$$
$$G_{43}(p)=G_{34}(p);\ G_{44}(p)=p_7+p_8+p_9;\ G_{45}(p)=-p_9;$$
$$G_{51}(p)=G_{15}(p);\ G_{54}(p)=G_{45}(p);\ G_{55}(p)=p_6+p_9+p_{11};$$

(33d) $I_1(p)=e_1p_1$; $I_2(p)=e_2p_2-e_5p_5$;
$I_3(p)=e_5p_5+e_7p_7$; $I_4(p)=-e_7p_7$.

4.1. Primary *p*-solution Formulation

By Theorem 3

$$L=\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix},$$

$t=(100\ \ 100\ \ 0\ \ 0\ \ -10\ \ 0\ \ 10\ \ 0\ \ 0\ 0\ \ 0)^T$.

For this example method MI converges in 5 iterations and yields the outer interval solution $\boldsymbol{x}$. Its first component is

(34) $\mathbf{x}_1^I = [55.10036, 66.93172]$

(here and in what follows interval solutions are rounded outwardly). Method MA converges also in 5 iterations to produce the p-solution sought

(35) $\mathbf{x}(p)=\check{x}+Lp+[-\hat{x}, \hat{x}]$, $\mathbf{p}\in\mathbf{p}$.

We give data for its first component

(35a) $\check{x}_1=61.016043$, $\hat{x}_1=0.980712$,

(35b) $L_{(1:)}=(1.751151\ 0.718065\ -0.122766\ -0.110727\ -0.169550\ -0.926535\ -0.010466\ -0.173983\ -0.070920\ -0.404930\ -0.479339)$

($L_{1:}$ denotes the first row of L). By the present method the outer interval solution x_1^A is obtained reducing the *p*-solution (35) to an interval. Hence

(36) $\mathbf{x}_1^A = [55.10036, 66.93172]$

and it is seen that for this particular example $\mathbf{x}_1^A=\mathbf{x}_1^I$. However, $\boldsymbol{x}^I$ and $\boldsymbol{x}^A$ are computed for times $t_1=0.121$s and $t_2=0.978$s, respectively, so method MA is about 8 times slower (in its present implementation) than method MI. It should, however, be stressed that unlike method MI the present method MA provides by Theorem 2 the two-sided bounds on $\underline{x}_k^*$ and $\overline{x}_k^*$. It has been obtained $\mathbf{e}_1^l=[\underline{e}_1^l, \bar{e}_1^l]=[55.10036, 56.07762]$. As is known [11] $\underline{x}_1^*=55.96681$ and it is seen that indeed $\underline{x}_1^*$ is in $\mathbf{e}_1^l$. In a similar way $\mathbf{e}_1^u=[\underline{e}_1^u, \bar{e}_1^u]=[65.95447, 66.93172]$, $\overline{x}_1^*=66.84634$, and also $\overline{x}_1^*$ is in $\mathbf{e}_1^u$.

Better bounds $\mathbf{e}_1^l$ and $\mathbf{e}_1^u$ are obtained if $\bar{e}_1^l$ and $\underline{e}_1^u$ are computed using Theorem 2. Now $\bar{e}_1^l=55.96681$ and it is seen that $\bar{e}_1^l=\underline{x}_1^*$; also $\underline{e}_1^u=66.84634=\overline{x}_1^*$. Thus, as expected the new bounds [55.10036, 55.96681] and [66.84634, 66.93172] are narrower than the former bounds [55.10036, 56.07762] and [65.95447, 66.93172].

Remark 4. It is claimed in [20], item (iii) of Theorem 1, that the solutions of systems (23c) provide the respective endpoints $\underline{x}_k^*$ and $\overline{x}_k^*$ of the hull solution. Indeed, for the example considered it is seen that $\bar{e}_1^l=\underline{x}_1^*$ and $\underline{e}_1^u=\overline{x}_1^*$. However, detailed analysis shows that this claim remains valid only if (for a fixed center) the radius $\hat{u}$ of the parameter vector $\boldsymbol{u}$ does not exceed some critical value $\hat{u}_{cr}$. Thus, for the present example it has been established numerically that $\hat{u}_{cr}$ is attained roughly for $\tau_{cr}=0.24$. The theoretical determination or assessment of $\hat{u}_{cr}$ is, for the time being, an open problem.

The non-zero conditions of Assumption 1 remain valid also for k=2 to k=4. However, for k=5, $l_{54}=0$ $l_{55}=0$. The remaining non-zero elements lead to the sign vector

$s^{l(1)}=(1\ \ 1\ \ -1\ \ 0\ \ 0\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1)$.

We only sketch the computation of the bounds $\mathbf{e}_1^l$ and $\mathbf{e}_1^u$ when formulae (23) are used. First $\bar{e}_5^l$ is determined. According to Procedure 1 we determine the vector $\underline{p}^{(1)}$ using (23a) with j≠4 and j≠5. Next we find the p-solution (22) of (25a) (where $m_2=2$) and see that the new $l_{54}<0$ and $l_{55}<0$. Thus we can find the respective two-dimensional sign vector $s^{l(1)}=(1\ \ 1)$ and the corresponding $\underline{p}^{(2)}$. Note that

$s^l=(1\ \ 1\ \ -1\ \ 1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ -1\ \ -1)$.

Next, the total vector $\underline{p}'=(\underline{p}^{(1)}, \underline{p}^{(2)})$ is formed and $\bar{e}_5^l$ is found as the solution of $A(\underline{p}')x=b(\underline{p}')$. Finally

$\mathbf{e}_5^l=[\underline{e}_5^l, \bar{e}_5^l]=[55.10036, 56.07762]$.

The end $\underline{e}_1^u$ is determined in a similar way using Procedure 2. Now

$s^{u(1)}=(-1\ \ -1\ \ 1\ \ 0\ \ 0\ \ -1\ \ -1\ \ 1\ \ 1\ \ 1\ \ 1)$

which leads to $\bar{p}^{(1)}$. Then we find he respective sign vector $s^{u(2)}=(1\ \ 1)$ and the corresponding $\bar{p}^{(2)}$ to form $\bar{p}'=(\bar{p}^{(1)}), \bar{p}^{(2)})$. Finally, $\underline{e}_1^u$ is the solution of $A(\bar{p}')x=b(\bar{p}')$. Thus, we have determined the bound

$e_5^u=[\underline{e}_5^u, \bar{e}_5^u]=[65.95447, 66.93172]$

Remark 5. If Assumption 1 is valid than from (23a) always $s^u=-s^l$. In the general case, this equality may be violated. Indeed, in this example $s^{u(2)}=s^{l(2)}$ so

$s^u=(-1\ \ -1\ \ 1\ \ 1\ \ 1\ \ -1\ \ -1\ \ 1\ \ 1\ \ 1\ \ 1)$

and it is seen that $s^u\neq -s^l$. This fact was experimentally established in [11] in a much harder way using global or local monotonicity conditions.

4.2. Secondary *p*-solution Formulation

As an example of such formulation, consider the problem of computing an outer solution for the current i_3. In this case we have

(37a) $z=i_3=(x_1-x_2)p_3'$.

Using the present paper's approach, we have to determine the p-solution of the secondary problem (33), (37a). This can be done computing

(37b) $\langle z\rangle=(\langle x_1\rangle-\langle x_2\rangle)\langle p_3'\rangle$.

Thus, the secondary p-solution is

(38a) $\check{z}$=0.043753, $\hat{z}$=0.016205,

(38b) L^z=(0.011160 -0.010529 0.002082 0.002237 0.003425 -0.006738 0.001110 0.000195 0.000851 0.004704 -0.002421).

To compare the efficiency of the two methods MI and MA we first reduce (38) to an interval

(39a) $\mathbf{i}_3^A=\mathbf{z}=[\underline{z}, \bar{z}]=[-0.01379, 0.10546]$.

We now compute the respective interval obtained by method MI using the formula $\mathbf{i}_3^I=(\mathbf{x}_1^I-\mathbf{x}_2^I)\mathbf{p}'_3$:

(39b) $\mathbf{i}_3^I=[-0.08459, 0.18085]$.

It should be stressed that, as seen from (39a) and (39b), the former interval is much narrower than the latter interval.

Next, we compute the bounds $\mathbf{e}_z^l$ and $\mathbf{e}_z^u$ on $\underline{z}^*$ and $\bar{z}^*$, respectively. Using $\bar{e}_z^l=\check{z}-\sum_j|l_j|$ and $\underline{e}_z^u=\check{z}+\sum_j|l_j|$ we have

$\mathbf{e}_z^l=[-0.01379, 0.00038245]$, $\mathbf{e}_z^u=[0.091288, 0.10546]$.

Finally, using (30a), (30b) slightly better results for $\bar{e}_z^l$ and $\underline{e}_z^u$ are computed: $\bar{e}_z^l=0.00041946$ and $\underline{e}_z^u=0.091166$. Thus, the bounds $\mathbf{e}_z^l$ and $\mathbf{e}_z^u$ on $\underline{z}^*$ and $\bar{z}^*$ are obtained

$\mathbf{e}_z^l=[-0.01379, 0.00041946]$, $\mathbf{e}_z^u=[0.091166, 0.10546]$.

Since the function in (37a) involves the nonlinear operation of multiplication, the vertex property of $\underline{z}^*$ and $\bar{z}^*$ cannot be guaranteed. Therefore, the interval $[\bar{e}_z^l, \underline{e}_z^u]$ only provides an inner estimation $z^{in}=[0.00041946, 0.091166]$ of the hull solution z^*.

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