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Q-expected Utility of *p*-approximated Generalized Lotteries of I Type Using Wald, Maximax and Hurwicz$_\alpha$ Criteria

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Key Words: *Ribbon distributions; interval estimates; lotteries; strict uncertainty; expected utility.*

Abstract. *Our focus is on one-dimensional fuzzy-rational generalized lotteries of I type, where the set of prizes is continuous, and the uncertainty is partially quantified by p-ribbon distribution functions (CDFs). The p-ribbon CDFs originate from the interval estimates of quantiles. Our objective is to rank such alternatives using several modifications of the expected utility rule. Initially, we transform the p-ribbon functions into classical ones using one of three decision criteria Q under strict uncertainty – Wald, maximax and Hurwicz$_\alpha$. That approximated the p-fuzzy-rational generalized lotteries of I type into classical pQ-generalized lotteries of I type. We can then calculate the Wald, maximax and Hurwicz$_\alpha$ expected utility to rank them. We prove that to find those expected utilities we need to estimate the inner quantile indices of the CDF in the pQ-generalized lotteries of I type. A universal algorithm to find the Wald-expected utility of a one-dimensional p-fuzzy-rational generalized lottery of I type is proposed, along with six simplified algorithms analyzing the cases when the utility function is either partially linearly interpolated or arctan approximated and also interprets different types of preferences (monotonic or non-monotonic). The maximax and Hurwicz$_\alpha$ expected utilities are then derived using trivial modifications of the procedures developed for the Wald expected utility. Two numerical examples demonstrate the application of the procedures*

1. Introduction

Risk is a key factor that leads to complexity in situations, where the decision maker (DM) must make a choice. There are several scenarios for decision making depending on the level of uncertainty.

The scenario under strict certainty is where the DM knows for sure which state of nature will occur, therefore there is a direct link between the choice of action and the outcome (that occurs with 100% chance). This setup is utilized to derive axioms of rational preference over alternatives, as well as to define various types of numerical functions that measure preferences. All findings are then applicable to the case under risk.

The scenario of strict uncertainty has been addressed by many authors to arrive at several decision criteria, among which the Wald [Fabrycky et al., 1998], Hurwicz$_\alpha$ [Yager, 2006], and the maximax criteria [Hackett, Luffrum, 1999]. The Wald criterion is an extreme pessimistic technique and assumes that alternatives should be ranked in descending order of their worst outcome. The maximax criterion is the opposite and ranks alternatives in descending order of their best outcomes. A criterion that balances both those extremes is the Hurwicz$_\alpha$ criterion. It finds the estimate of a pessimism index $\alpha\epsilon[0;1]$, such that the higher α is the more pessimistic the DM. It then ranks the alternatives in descending order of the value of the worst and best prizes weighted by α, i.e. taking into account the actual pessimistic/optimistic profile of the DM. Typically, α is estimated once for a given DM, but a more adequate approach is to define it for each decision problem. As discussed in [French, Insua, 2010], none of the criteria under strict uncertainty obey the minimal requirements of rational choice.

Decisions in the case of uncertainty are discussed in detail in [Etner et al., 2009]. The best-known and widely used technique for rational decisions under risk is the utility theory of Von Neumann and Morgenstern (1947). It offers the lottery model as an adequate interpretation of alternatives under risk. Lotteries are described as a full disjoint set of events each one of those giving a prize. When the set of prizes is finite, we introduce ordinary lotteries. When the set of prizes and/or the set of lotteries are continuous, then generalized lotteries of various types are introduced [Nikolova et al., 2010a]. Our paper shall focus on generalized lotteries of I type (GL-I) in the case of one-dimensional (1-D) prizes probabilistically described by a cumulative distribution function (CDF).

The uncertainty associated with the events in the lotteries is measured by probabilities, whereas the utility function *u*(.) measures how much the decision maker likes each of the prizes. The utility function *u*(.) is constructed so that it increases with the increase of preferences of the decision maker (DM) [Keeney, Raiffa, 1993]. Once probabilities and utilities are available, it is possible to calculate the expected utility of each lottery and rank them in descending order of this index [French, Insua, 2010].

The classical methods for elicitation suggest that the DM should identify unique probability and utility estimates. However, the real DM has limited ability to differentiate between close alternatives and can only elicit probabilities and utilities in an interval form. Subsequently, some of the axioms of rationality of choice do not hold, which is why [Nikolova, et al., 2005] discussed fuzzy-rational DMs. The interval nature of probabilities leads to the necessity to use ribbon distributions to describe uncertainty. For a 1-D random variable, the CDF is then either *x*-ribbon or *p*-ribbon depending on the type of unquantified uncertainty (on the quantile or on the quantile index) [Tenekedjiev, et al., 2006]. As a result, risky alternatives are represented as fuzzy-rational lotteries and cannot be directly ranked according to expected utility. What is possible is to approximate the ribbon distribution by a classical one using some of the above-mentioned criteria under strict uncertainty. What makes those criteria suitable is the essence of interval estimates. The true value of a given elicited parameter (either utility or probability) is any value between the lower and upper bound of its uncertainty interval. However, there is no way to know which one it is, with each value in the interval having the potential to be the true one. This replicates the case under strict uncertainty, where the states of nature may be defined, but no information is available to define their likelihoods. This justifies the use of decision criteria under strict uncertainty as a systematic way to approximate ribbon distributions.

The process to rank fuzzy-rational GL-I with one-dimensional ribbon distribution functions was discussed in detail in [Tenekedjiev, et al., 2006]. The works [Nikolova, 2007; Nikolova, Tenekedjieva, 2007] applied the Hurwicz$_\alpha$ criterion to rank fuzzy-rational GL-I with *x*-ribbon distributions. The Hurwicz$_\alpha$, Wald and maximax criteria were also utilized in [Tenekedjiev, 2006] to rank fuzzy-rational ordinary lotteries with p-ribbon distributions.

The objective of this paper is to apply the Wald, maximax and Hurwicz$_\alpha$ criteria under strict uncertainty to rank fuzzy-rational GL-I, where uncertainty is quantified using p-ribbon functions. We give a clear definition of the general problem setup of ranking fuzzy-rational GL-I with p-ribbon functions. In addition to defining and solving the general problem of ranking fuzzy-rational lotteries, we also discuss several special cases for the various forms of the utility function (monotonic, non-monotonic, increasing or decreasing) and the way the utility function is constructed (either partially linearly interpolated or arctan-approximated). We also present two empirical examples to demonstrate the application of the algorithms and procedures. They present a case with partially linearly interpolated and with arctan-approximated utility for non-monotonic and for decreasing preferences.

2. Description of Uncertainty Using Probability Distributions

We typically quantify the uncertainty associated with continuous random variables *X* using various types of probability distributions. Their form and definition would change depending on whether we can entirely quantify the uncertainty, or we measure it based on interval estimates.

2.1. Probability Distributions when Uncertainty is Entirely Quantified – the Classical Case

Assume that we have entirely quantified the uncertainty associated with a one-dimensional (1D) random variable X using a known 1D cumulative distribution function $F(.)$, referred to as classical CDF and defined for $x \in (-\infty;+\infty)$. Then for a given value x of X we know that

(1) $F(x)=P(X\leq x)$ for $x \in (-\infty;+\infty)$.

The most convenient and often utilized approach to construct such classical distribution functions is through partially linear interpolation using a set of z>1 preliminarily chosen points from the function that obey the following conditions:

(2) $\{(x_l\,;\,F_l)\mid l=1,2,\ldots,z\}$, where

$$x_1 \leq x_2 \leq \ldots \leq x_z\,,$$
$$0 = F_1 \leq F_2 \leq \ldots \leq F_z = 1\,.$$

Each pair (x_l, F_l) is called a *node*, where x_l is the α -quantile of the random variable X with $\alpha=F_l$ (for $l=1,2,\ldots,z$). Then partially linear interpolation may be performed as follows:

$$F(x)= \tag{3}$$
$$=\begin{cases} 0 & \text{for } x < x_1 \\ F_i & \text{for } x_i = x < x_{i+1},\ i=1,2,\ldots,z-1 \\ F_i + \dfrac{(x-x_i)(F_{i+1}-F_i)}{x_{i+1}-x_i} & \text{for } x_i < x < x_{i+1},\ i=1,2,\ldots,z-1 \\ 1 & \text{for } x_z \leq x \end{cases}$$

2.2. Probability Distributions when Uncertainty is Partially Quantified

Let the uncertainty in a 1-D random variable X be partially quantified by a 1-D distribution function $F^R(.)$. It is only known that it entirely lies between the so called *lower* and *upper border functions* $F_d(.)$ and $F_u(.)$, i.e.

(4) $F_d(x) \leq F^R(x) \leq F_u(x)$, for $x \in (-\infty;+\infty)$.

Here, $F_d(.)$ and $F_u(.)$ are known classical distribution functions, which obey the condition

(5) $F_d(x) \leq F_u(x)$, for $x\in(-\infty;+\infty)$.

A 1-D distribution function $F^R(.)$ that obeys this definition shall be called *ribbon distribution function*.

2.2.1. 1-D x-ribbon Distribution Functions

A common special case is to construct distributions (usually subjective) by interpolation on nodes with an uncertainty interval for the quantile (error on the abscissa x). Then the fuzzy distribution function may be called *x-ribbon* $F^{xR}(.)$, whereas the border functions are respectively *lower* and *upper x-border functions* $F_{xd}(.)$ and $F_{xu}(.)$.

A convenient way to define x-border distribution functions is via partially linear interpolation on the margins of the set of z>1 defined uncertainty intervals for quantiles of the x-ribbon function:

(6) $\{(x_{d,i}\,;\,x_{u,i}\,;\,F_i)\mid i=1,2,\ldots,z\}$, where

$$x_{d,1} \leq x_{d,2} \leq \ldots \leq x_{d,z}\,,$$
$$x_{u,1} \leq x_{u,2} \leq \ldots \leq x_{u,z}\,,$$
$$x_{d,i} \leq x_{u,i}\,,\ \text{for } i=2,3,\ldots,z-1,$$
$$x_{d,1} = x_{u,1},\ \ x_{d,z} = x_{u,z}\,,$$
$$0 = F_1 \leq F_2 \leq \ldots \leq F_z = 1\,.$$

Then

(7)

$$F_{xd}\left(x\right)= \\ =\begin{cases} 0 & \text{for } x < x_{d,1} \\ F_i & \text{for } x_{d,i} = x < x_{d,i+1}, \\ & i\text{=}1,2,...,z-1 \\ F_i + \dfrac{\left(x - x_{d,i}\right)\left(F_{i+1} - F\right)}{x_{d,i+1} - x_{d,i}} & \text{for } x_{d,i} < x < x_{d,i+1}, \\ & i\text{=}1,2,...,z-1 \\ 1 & \text{for } x_{d,z} \le x \end{cases}$$

(8)

$$F_{xu}\left(x\right)= \\ =\begin{cases} 0 & \text{for } x < x_{u,1} \\ F_i & \text{for } x_{u,i} = x < x_{u,i+1}, \\ & i\text{=}1,2,...,z-1 \\ F_i + \dfrac{\left(x - x_{u,i}\right)\left(F_{i+1} - F_i\right)}{x_{u,i+1} - x_{u,i}} & \text{for } x_{u,i} < x < x_{u,i+1}, \\ & i\text{=}1,2,...,z-1 \\ 1 & \text{for } x_{u,z} \le x \end{cases}$$

(9) $F_{xd}(x) \le F^{xR}(x) \le F_{xu}(x)$, for $x \in (-\infty; +\infty)$.

2.2.2. 1-D p-ribbon Distribution Functions

Another common approach is when (usually subjective) distributions are interpolated on nodes with uncertainty interval for the quantile index (error on the ordinate, i.e. probability). Then the fuzzy distribution function may be called *p-ribbon* $F^{pR}(.)$, whereas the border functions are respectively *lower* and *upper p-border functions* $F_{pd}(.)$ and $F_{pu}(.)$.

A convenient way to define p-border distribution functions is by partially linear interpolation on the borders of the set of z>1 defined uncertainty intervals for quantile indices of the p-ribbon function:

(10) $\{(x_i\,;\,F_{d,i}\,;\,F_{u,i}) \mid i\text{=}1,2,\ldots,z\}$, where

$$x_1 \le x_2 \le ... \le x_z\,,$$

$$0 = F_{d,1} \le F_{d,2} \le ... \le F_{d,z} = 1\,,$$

$$0 = F_{u,1} \le F_{u,2} \le ... \le F_{u,z} = 1\,,$$

$F_{d,i} \le F_{u,i}$, for i=2, 3, …, z–1.

Then

(11)

$$F_{pd}\left(x\right)== \\ \begin{cases} 0 & \text{for } x < x_1 \\ F_{d,i} & \text{for } x_i = x < x_{i+1}, \\ & i\text{=}1,2,...,z-1 \\ F_{d,i} + \dfrac{\left(x - x_i\right)\left(F_{d,i+1} - F_{d,i}\right)}{x_{i+1} - x_i} & \text{for } x_i < x < x_{i+1}, \\ & i\text{=}1,2,...,z-1 \\ 1 & \text{for } x_z \le x \end{cases}$$

(12)

$$F_{pu}\left(x\right)= \\ =\begin{cases} 0 & \text{for } x < x_1 \\ F_{u,i} & \text{for } x_i = x < x_{i+1}, \\ & i\text{=}1,2,...,z-1 \\ F_{u,i} + \dfrac{\left(x - x_i\right)\left(F_{u,i+1} - F_{u,i}\right)}{x_{i+1} - x_i} & \text{for } x_i < x < x_{i+1}, \\ & i\text{=}1,2,...,z-1 \\ 1 & \text{for } x_z \le x \end{cases}$$

(13) $F_{pd}(x) \le F^{pR}(x) \le F_{pu}(x)$, for $x \in (-\infty; +\infty)$.

3. Ranking Alternatives Modeled as 1-D Generalized Lotteries of I Type

The problem, which we face is to select the best alternative out of a set of alternatives L a.k.a lottery set. Let for each alternative in L the prize $\vec{x}$ is a realization of continuous or mixed random variable X described with its CDF. If L is countable then the elements of L are called generalized lotteries of first type GL-I. To achieve rational ranking of alternatives we need to build the utility function $u(.)$ over all possible values of X for all alternatives in L. As mentioned in the introduction the alternatives should be ranked according to the expected utility of each alternative. In this paper we will deal with 1D prizes, so the CDF of each alternative in L will be 1D too.

3.1. One-dimensional Generalized Lotteries of I Type with Entirely Quantified Uncertainty

Let's compare q alternatives according to DM's preference. Each alternative gives 1-D prizes x with a utility function $u(.)$, defined for all possible prizes from all alternatives.

A 1-D generalized lottery of I type with a classical distribution function $F_k(.)$ shall be called *classical risky generalized lottery of I type*:

(14) gk=<F_k(x) ; x >, for k=1,2,…, q.

A theorem [French, Insua, 2010] proves that such lotteries should be ranked in descending order of the expected utility, which is calculated as a Stieltjes integral:

(15) $$E_k\left(u \mid F_k\right) = \int_{-\infty}^{+\infty} u\left(x\right) dF_k\left(x\right).$$

Let us have a special case, when $F_k(.)$ is a partially linear distribution function with nodes

(16) $$\left\{\left(x_i^{(k)}; F_i^{(k)}\right) \mid i = 1,\, 2, \ldots,\, z_k\right\}, \text{ where}$$

$$x_1^{(k)} \le x_2^{(k)} \le ... \le x_{z_k}^{(k)}\,,$$

$$0 = F_1^{(k)} \le F_2^{(k)} \le ... \le F_{z_k}^{(k)} = 1\,.$$

This is the most common approach to construct $F(.)$ and practically the one utilized when solving decision tasks. Then the expected utility is brought down to the following [Nikolova et al., 2010a]:

$$E_k\left(u \mid F_k\right)=\int_{-\infty}^{+\infty} u(x) dF_k(x)=$$

$$(17)\quad = \sum_{\substack{i=1 \\ x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} u(x) \mathrm{d}x + \sum_{\substack{i=1 \\ x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right) u\left(x_i^{(k)}\right)$$

Here, the integral under the first summation is a Riemann integral and causes no problems. The solution (17) works equally well for continuous r.v and for mixed r.v.

There are two special cases for the description of utility, which dominate in practice.

3.1.1. Partially Linear-Interpolated Utility

A common way to construct the utility function is through partially linear interpolation on a set of $z^u>1$ nodes containing only point estimates:

$$(18)\quad \left|\begin{array}{l} \left\{\left(x_j^u ; u_j^u\right) \mid j=1,2, \ldots, z^u\right\}, \\ x_d \equiv x_1^u<x_2^u<\ldots<x_{z^u}^u \equiv x_u . \\ x_d \leq x_1^{(k)} \text { and } x_{z_k}^{(k)} \leq x_u \quad \text { for } k=1,2, \ldots, q \end{array}\right.$$

The resulting partially linearly interpolated utility function, whose domain is the closed interval $[x_d; x_u]$, takes the form:

$$(19)\quad u(x)= \begin{cases}u_j^u & x=x_j^u, \quad j=1,2, \ldots, z^u \\ u_j^u+\left(x-x_j^u\right) \dfrac{\left(u_{j+1}^u-u_j^u\right)}{\left(x_{j+1}^u-x_j^u\right)} & x_j^u<x<x_{j+1}^u, \quad j=1,2, \ldots, z^u-1\end{cases}$$

Since the CDF is partially linearly interpolated on the nodes (16) and the utility is partially linearly interpolated on the nodes (18) we can add additional nodes which lie on the initial lines and the functions will not change. Using the stated property an algorithm has been provided in [Nikolova et al., 2010b] to create a set of $z^{(k)g}$ triplets:

(20) $\left(x_i^{(k)g} ; F_i^{(k)g} ; u_i^{(k)g}\right)$, for $i=1, 2, \ldots, z^{(k)g}$.

Here, the utility function and the distribution function are interpolated on nodes with the same abscissas. In the same source it is proven that the expected utility (17) can be simplified to

$$(21)\quad E_k\left(u \mid F_k\right)=\frac{1}{2} \sum_{i=1}^{z^{(k)g}-1}\left(F_{i+1}^{(k)g}-F_i^{(k)g}\right)\left(u_i^{(k)g}+u_{i+1}^{(k)g}\right)$$

3.1.2. Arctan Approximated Utility

One possible way to construct the utility function once data nodes are available is through approximation with a given analytical form. Many such forms can be found in [Keeney, Raiffa, 1993]. This is suitable in cases with a small number of elicited nodes and/or wide uncertainty intervals. The work [Nikolova, 2007] proposed a form for the case of strictly increasing preferences:

$$(22)\quad u(x)=\frac{\arctan\left(a x-a x_0\right)-\arctan\left(a x_d-a x_0\right)}{\arctan\left(a x_u-a x_0\right)-\arctan\left(a x_d-a x_0\right)}$$

Here, arctan(.) is the inverse tangent function. The end nodes do not contain error and are $u(x_d)=0$, $u(x_u)=1$. This form is proven to suit to prize sets containing both gains and losses. The parameter $a>0$ relates to the risk sensitivity of the DM, whereas x_0 is the inflex point in $u(.)$, which divides the interval of prizes into two sections – risk prone section (to the left), and risk averse section (to the right).

The work [Nikolova, 2007] proposed a form for the case of strictly decreasing preferences:

$$(23)\quad u(x)=\frac{\arctan\left(a x_u-a x_0\right)-\arctan\left(a x-a x_0\right)}{\arctan\left(a x_u-a x_0\right)-\arctan\left(a x_d-a x_0\right)}$$

The parameter $a>0$ again relates to the risk sensitivity of the DM, whereas x_0 is the inflex point in $u(.)$, which divides the interval of prizes into two sections – risk averse section (to the left), and risk prone section (to the right).

In [Nikolova et al., 2009] several reasons have been outlined to use (22) or (23) for the case of strictly monotonic preferences when the nodes of the utility function are assessed in an interval form (almost always). The unknown parameters a and x_0 can be identified by x^2-minimization [Nikolova, 2007] using the weighted least square method [Press et al., 2007].

For the case of strictly increasing utility, (22) may be represented as

(24) $u(x) = A_{incr}\arctan(ax - ax_0) + B_{incr}$

where

$$(25)\quad \begin{aligned} A_{incr} &= \frac{1}{\arctan\left(a x_u-a x_0\right)-\arctan\left(a x_d-a x_0\right)} \\ B_{incr} &= -\arctan\left(a x_d-a x_0\right) A_{incr} \end{aligned}$$

For the case of strictly decreasing utility, (23) may be represented as

(26) $u(x) = A_{decr}\arctan(ax - ax_0) + B_{decr}$

where

$$(27)\quad \begin{aligned} A_{decr} &= -\frac{1}{\arctan\left(a x_u-a x_0\right)-\arctan\left(a x_d-a x_0\right)} \\ B_{decr} &= -\arctan\left(a x_u-a x_0\right) A_{decr} \end{aligned}$$

It is convenient to unite (24) and (26) into a general form for the acrtan-approximated utility when preferences are monotonic:

(28) $u(x) = A \times \arctan(ax - ax_0) + B$.

After substituting (28) in (17) it is possible to derive a closed formula for the expected utility integral of GL-I for the case of arctan approximated utility with partially linearly interpolated CDF:

$$E_k(u \mid F_k) = A \sum_{\substack{i=1 \\ x_i^{(k)} < x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)} - F_i^{(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} C_i^{(k)} + \tag{29}$$
$$+A \sum_{\substack{i=1 \\ x_i^{(k)} = x_{i+1}^{(k)}}}^{z_k-1} (F_{i+1}^{(k)} - F_i^{(k)}) \arctan(ax_i^{(k)} - ax_0) + B$$

where

$$C_i^{(k)} = (x_{i+1}^{(k)} - x_0) \arctan(ax_{i+1}^{(k)} - ax_0) - \tag{30}$$
$$-(x_i^{(k)} - x_0) \arctan(ax_i^{(k)} - ax_0) +$$
$$+\frac{1}{2a} \ln \frac{1 + (ax_i^{(k)} - ax_0)^2}{1 + (ax_{i+1}^{(k)} - ax_0)^2}$$

Formulae (29) and (30) are proven in the Appendix. The above formulae were coined in [Nikolova et al., 2010b] but without formal proof and just for the case of increasing preferences.

3.2. One-dimensional Generalized Lotteries of First Type with Partially Quantified Uncertainty

3.2.1. Basic Definitions

A one-dimensional GL-I with a one-dimensional ribbon CDF shall be referred to as a *one-dimensional fuzzy rational GL-I*:

(31) $g_k^{fr} =< F_k^R(x); x >$, $k=1, 2, \ldots, q$.

Here $F_k^R(.)$ is a one-dimensional ribbon CDF with lower and upper distributional bounds $F_k^d(.)$ and $F_k^u(.)$.

1-D fuzzy-rational generalized lotteries of I type may be ranked at two stages:

1) Using a Q criterion under strict uncertainty, each one-dimensional ribbon CDF $F_k^R(.)$ is approximated by a one-dimensional classical CDF $F_k^Q(.)$, which obeys the following condition for all x $\in$ $(-\infty;+\infty)$

(32) $F_k^d(x) \le F_k^Q(x) \le F_k^u(x)$, $k=1, 2, \ldots, q$.

In that way each one-dimensional fuzzy rational GL-I is approximated by a one-dimensional classical risky GL-I, that shall be referred to as *Q-generalized (one-dimensional Q-GL-I)*,

(33) $g_k^Q =< F_k^Q(x); x >$ for $k=1,2,\ldots,q$.

2) The alternatives are ranked in descending order of the expected utilities of the one-dimensional Q-GL-I, using (15)

(34) $E_k^Q(u \mid F_k^R) = \int_{-\infty}^{+\infty} u(x) dF_k^Q(x)$, for $k=1,2,\ldots,q$.

Equation (34) uses a Stieltjes integral with an integrating function $F_k^Q(.)$.

The resulting criterion to rank one-dimensional fuzzy rational GL-I shall be called *Q-expected utility*. In this paper we will use 3 criteria for strict uncertainty – Wald [Fabrycky et al., 1998], maximax [Hackett, Luffrum, 1999] and Hurwicz$_\alpha$ [Yager, 2006]. The approximation of $F_k^R(.)$ by $F_k^Q(.)$ for the discussed criteria under strict uncertainty essentially relies on the availability of one-dimensional utility function $u(.)$ defined over all values of X.

Our paper will deal only with p-ribbon CDFs.

3.2.2. Problem Setup for Fuzzy-Rational GL-I with p-ribbon CDF

A one-dimensional fuzzy rational GL-I with a p-ribbon CDF shall be called one-dimensional p-fuzzy rational GL-I:

(35) $g_k^{pfr} =< F_k^{pR}(x); x >$, $k=1, 2,\ldots, q$.

Here $F_k^{pR}(x)$ is a one-dimensional p-ribbon CDF, whose lower and upper p-distributional bounds are $F_k^{pd}(.)$ and $F_k^{pu}(.)$. The latter are defined similarly to (11)-(13).

Calculating the Q-expected utility of the one-dimensional p-fuzzy rational GL-I may be brought down to the following steps:

1) Using a Q criterion under strict uncertainty, the one-dimensional p-ribbon CDF $F_k^{pR}(x)$ is piece-wise partially linearly approximated by a one-dimensional classical CDF $F_k^{pQ}(x)$ with nodes

(36) $\left\{\left(x_i^{(k)}; F_i^{Q,(k)}\right) \mid k=1,2,\ldots,z_k\right\}$,

where

(37)
$$\begin{cases} 0 = F_1^{Q,(k)} \le F_2^{Q,(k)} \le \ldots \le F_{z_k}^{Q,(k)} = 1, \\ F_i^{d,(k)} \le F_i^{Q,(k)} \le F_i^{u,(k)}, \ k=2, 3,\ldots, z_k-1. \end{cases}$$

Then

(38)
$$F_k^{pR}(x) = \begin{cases} 0 & \text{for } x < x_1^{(k)}, \\ F_i^{Q,(k)} & \text{for } x_i^{(k)} = x < x_{i+1}^{(k)}, \\ & i=1, 2, \ldots, z_k-1, \\ F_i^{Q,(k)} + \\ +\dfrac{\left(x - x_i^{(k)}\right)\left(F_{i+1}^{Q,(k)} - F_i^{Q,(k)}\right)}{x_{i+1}^{(k)} - x_i^{(k)}} & \text{for } x_i^{(k)} < x < x_{i+1}^{(k)}, \\ & i=1, 2, \ldots, z_k-1, \\ 1 & \text{for } x_{z_k}^{(k)} \le x. \end{cases}$$

In that way the one-dimensional p-fuzzy rational GL-I is approximated by a one-dimensional classical risky GL-I, which shall be referred to as *pQ-generalized (one-dimensional pQ-GL-I)*,

(39) $g_k^{pQ} = < F_k^{pQ}(x); x >$.

2) The Q-expected utility of the one-dimensional *p*-fuzzy rational GL-I is calculated as the expected utility of the one-dimensional *p*Q-GL-I using formula (17):

(40)
$$E_k^{pQ}(u \mid F_k^{pR}) = \int_{x_1^{(k)}}^{x_{z_k}^{(k)}} u(x)\, dF_k^{pQ}(x) = \sum_{\substack{i=1 \\ x_{k+1} > x_k}}^{z_k-1} \frac{F_{i+1}^{Q,(k)} - F_i^{Q,(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} u(x)\, dx + \sum_{\substack{i=1 \\ x_{i+1} = x_i^{(k)}}}^{z_k-1} \left(F_{i+1}^{Q,(k)} - F_i^{Q,(k)} \right) u\left(x_i^{(k)}\right).$$

Here the integral under the first summation symbol is a Riemann integral.

The resulting *Q*-expected utility of the one-dimensional *p*-fuzzy rational GL-I shall be called *pQ-expected utility*. The application of some of the criteria under strict uncertainty essentially relies on the one-dimensional utility function *u*(.) when approximating $F_k^{pR}(x)$ by $F_k^{pQ}(x)$.

So, in the special case of one-dimensional *p*-fuzzy rational GL-I, calculating the *pQ*-expected utility of the i-th fuzzy rational GL-I is brought down to the estimation of the inner quantile indices $F_i^{Q,(k)}(x)$, i=2, 3,…, z_k–1, of the classical CDF in the pQ-GL-I g_k^{pQ}. The problem may be formalized as follows:

General problem

Given:

– criterion under strict uncertainty Q;

– one-dimensional utility function *u*(.);

– number of approximating nodes $z_k > 1$;

– quantiles $x_i^{(x)}$, k=1, 2,…, z_k, such that

(41) $x_1^{(k)} \le x_2^{(k)} \le \ldots \le x_{z_k}^{(k)}$;

– lower quantile index bounds $F_i^{d,(k)}$, i=1, 2,…, z_k , such that

(42) $0 = F_1^{d,(k)} \le F_2^{d,(k)} \le \ldots \le F_{z_k-1}^{d,(k)} \le F_{z_k}^{d,(k)} = 1$;

– upper quantile index bounds $F_i^{u,(k)}$, i=1, 2,…, z_k, such that

(43) $0 = F_1^{u,(k)} \le F_2^{u,(k)} \le \ldots \le F_{z_k-1}^{u,(k)} \le F_{z_k}^{u,(k)} = 1$;

– end quantile indices

(44) $F_1^{Q,(k)} = F_1^{d,(k)} = F_1^{u,(k)} = 0$,

(45) $F_{z_k}^{Q,(k)} = F_{z_k}^{d,(k)} = F_{z_k}^{u,(k)} = 1$.

Find:

– inner quantile indices $F_i^{Q,(k)}$, i=2, 3, …, z_k–1, such that

(46) $F_2^{Q,(k)} \le F_3^{Q,(k)} \le \ldots \le F_{z_k-2}^{Q,(k)} \le F_{z_k-1}^{Q,(k)}$,

(47) $F_i^{d,(k)} \le F_i^{Q,(k)} \le F_i^{u,(k)}$.

– the *Q*-expected utility $E_k^{pQ}(u|F_k^{pR})$ of the one-dimensional p-fuzzy rational GL-I g_k^{pQ}.

We will provide three solutions of Problem 1 depending on the *Q* criterion.

3.2.2. Solution of the General Problem Using the Wald Criterion (Q=W)

The Wald decision criterion under strict uncertainty assumes that the worst outcome always occurs [Fabrycky et al., 1998]. The application of that concept in the case of a one-dimensional p-fuzzy rational GL-I implies to choose the quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1, so that to minimize the *pW*-expected utility of the lottery given in (89). Let us introduce (z_k–1) auxiliary variables:

(48)
$$I_i^{p,(k)} = \begin{cases} \dfrac{1}{x_{i+1}^{(k)} - x_i^{(k)}} \displaystyle\int_{x_i^{(k)}}^{x_{i+1}^{(k)}} u(x)\, dx & \text{for } x_{i+1}^{(k)} > x_i^{(k)}, \\ u\left(x_i^{(k)}\right) & \text{for } x_{i+1}^{(k)} = x_i^{(k)}. \end{cases}$$
for *i*=1,2, 3,…, z_k–1,

The variables $I_i^{p,(k)}$ are the mean utility in the closed interval $[x_i^{(k)}; x_{i+1}^{(k)}]$. That is why $I_i^{p,(k)}$ can be referred to as the *i*-th mean utility. The integral in the first line is a Riemann one.

The *pW*-expected utility of the *k*-th 1-D p-fuzzy rational GL-I would be rearranged as a constant plus a linear combination of the unknown quantile indices $F_i^{W,(k)}$, using the mean utilities (48):

(49)
$$E_k^{pW}(u \mid F_k^{pR}) = \int_{x_1^{(k)}}^{x_{z_k}^{(k)}} u(x)\, dF_k^{pW}(x) =$$
$$= \sum_{\substack{i=1 \\ x_{k+1} > x_k}}^{z_k-1} \frac{F_{i+1}^{W,(k)} - F_i^{W,(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} u(x)\, dx + \sum_{\substack{i=1 \\ x_{i+1} = x_i^{(k)}}}^{z_k-1} \left(F_{i+1}^{W,(k)} - F_i^{W,(k)} \right) u\left(x_i^{(k)}\right) =$$
$$= -F_1^{W,(k)} I_1^{p,(k)} + \sum_{i=2}^{z_k-1} F_i^{W,(k)} \left(I_{i-1}^{p,(k)} - I_i^{p,(k)} \right) + F_{z_k}^{W,(k)} I_{z_k-1}^{p,(k)} = -(0) I_1^{p,(k)} + \sum_{i=2}^{z_k-1} F_i^{W,(k)} \left(I_{i-1}^{p,(k)} - I_i^{p,(k)} \right) +$$
$$+ (1) I_{z_k-1}^{p,(k)} = I_{z_k-1}^{p,(k)} + \sum_{i=2}^{z_k-1} F_i^{W,(k)} \left(I_{i-1}^{p,(k)} - I_i^{p,(k)} \right)$$

Formula (49) considers (44) and (45), i.e. that $F_l^{W,(k)} = 0$ and $F_{z_k}^{W,(k)} = 1$.

So, the required quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1, may be identified by solving the following linear programming (LP) problem in canonical form: minimize the linear function

(50) $$\sum_{i=2}^{z_k-1} F_i^{W,(k)}\left(I_{i-1}^{p,(k)} - I_i^{p,(k)}\right) \text{ on } F_2^{W,(k)}, F_3^{W,(k)}, \ldots, F_{z_k-1}^{W,(k)}$$

provided the following $3z_k$–5 linear constraints hold:

(51) $$\begin{cases} F_i^{d,(k)} - F_i^{W,(k)} \le 0,\ i=2, 3, \ldots, z_k-1, \\ F_i^{W,(k)} - F_i^{u,(k)} \le 0,\ i=2, 3, \ldots, z_k-1, \\ F_i^{W,(k)} - F_{i+1}^{W,(k)} \le 0,\ i=2, 3, \ldots, z_k-2, \\ -F_2^{W,(k)} \le 0, \\ F_{z_k-1}^{W,(k)} - 1 \le 0. \end{cases}$$

We propose to solve the formulated LP problem (50)-(51) utilizing resources from MATLAB, i.e. the function *linprog* [The MathWorks, 2018] (refer to the Conclusion section for more detail on the use of Matlab in this study). The *linprog* function uses the Dual-Simplex Algorithm [Koberstein, 2008; Nocedal, Wright, 2006] to solve the LP task.

After finding the inner quantiles, the *W*-expected utility may be easily calculated using (49):

(52) $$E_k^{pW}(u \mid F_k^{pR}) = I_{z_k-1}^{p,(k)} + \sum_{i=2}^{z_k-1} F_i^{W,(k)}\left(I_{i-1}^{p,(k)} - I_i^{p,(k)}\right)$$

So, the General Problem stated can be universally solved using the Wald criterion using the following algorithm:

Universal Algorithm for finding the Wald-Expected utility of a 1D p-fuzzy rational GL-I

1) Calculate the mean utilities $I_i^{p,(k)}$ for i=1, 2,…, z_k-1 using (48) by numerically integrating the Riemann integral in the upper line.

2) Solve the LP problem (50) under the linear constraints (51) and find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly approximated classical CDF F_k^{pW}.

3) Calculate the Wald-expected utility of the 1D *p*-fuzzy rational GL-I using (52)

The Universal Algorithm under Wald can be simplified in some special cases depending on the utility function.

3.2.3.1. Partially Linear-Interpolated Non-Monotonic Utility under Wald Criterion (Q=W)

Let the utility (19) be non-monotonic and partially linear-interpolated on the nodes (18). Then the integral in (49) can be easily calculated using the approximation (20). Let i_{beg} and i_{end} are such indices that:

(53) $$x_{i_{beg}}^{(k)g} = x_i^{(k)} \text{ and } x_{i_{end}}^{(k)g} = x_{i+1}^{(k)}$$

Then the mean utilities (50) can be found in close form by

(54) $$I_i^{p,(k)} = \begin{cases} \dfrac{1}{2\left(x_{i+1}^{(k)} - x_i^{(k)}\right)} \times \\ \times \displaystyle\sum_{i=i_{beg}}^{i_{end}-1} \left(x_{i+1}^{(k)g} - x_i^{(k)g}\right)\left(u_{i+1}^{(k)g} + u_i^{(k)g}\right) & \text{for } x_{i+1}^{(k)} > x_i^{(k)}, \\ u(x_i^{(k)}) & \text{for } x_{i+1}^{(k)} = x_i^{(k)}. \end{cases}$$

So, in the special case of partially linear-interpolated utility the General Problem in 3.2.2 can be solved using the Wald criterion utilizing the following simplification of the Universal Algorithm:

Simplified Algorithm 1

1) Create the $z^{(k)g}$ triplets ($x_i^{(k)g}$; $F_i^{(k)g}$; $u_i^{(k)g}$) leaving the $F_i^{(k)g}$ unknown.

2) Calculate the mean utilities $I_i^{p,(k)}$ for *i*=1, 2,…, z_k-1 using (53) and (54).

3) Solve the LP problem (50) under the linear constraints (51) and find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly approximated classical CDF F_k^{pW}.

4) Calculate the Wald-expected utility of the 1D *p*-fuzzy rational GL-I using (52).

3.2.3.2. Increasing Mean Utilities under Wald Criterion (Q=W)

Let the mean utilities be increasing for g_k^{pfr}:

(55) $$I_{i-1}^{p,(k)} \le I_i^{p,(k)},\ i=2, 3, \ldots, z_k-1.$$

Condition (55) is often fulfilled, e.g. when the utility function *u*(.) is monotonically increasing in the interval [$x_l^{(k)}$; $x_{z_k}^{(k)}$] so that for $x_a \in [x_l^{(k)}; x_{z_k}^{(k)}]$ and $x_b \in [x_l^{(k)}; x_{z_k}^{(k)}]$

(56) if $x_a > x_b$, then $u(x_a) \ge u(x_b)$.

Since the coefficients of the linear function $\sum_{i=2}^{z_k-1} F_i^{W,(k)}\left(I_{i-1}^{p,(k)} - I_i^{p,(k)}\right)$ are entirely non-positive, then the minimum would be identified only for the greatest values of the unknown variables that obey the linear constraints, i.e.

(57) $$F_k^{W,(k)} = F_k^{u,(k)}, \qquad i=2, 3, \ldots, z_k-1.$$

So, in the special case of increasing mean utilities the General Problem in 3.2.2 can be solved using the Wald criterion utilizing the following simplification of the Universal Algorithm:

Simplified Algorithm 2

1) Calculate the mean utilities $I_i^{p,(k)}$ for *i*=1, 2,…, z_k-1 using (48) by numerically integrating the Riemann integral in the upper line.

2) Confirm the condition (55) and use (57) to find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly interpolated classical CDF F_k^{pW}.

3) Calculate the Wald-expected utility of the 1D p-fuzzy rational GL-I using (52).

3.2.3.3. Decreasing Mean Utilities under Wald Criterion (Q=W)

Let the mean utilities be decreasing for g_k^{pfr}:

(58) $I_{i\text{-}1}^{p,(k)} \geq I_i^{p,(k)}$, i=2, 3, ..., z_k–1.

Condition (58) is often fulfilled, e.g. when the utility function u(.) is monotonically decreasing in the interval $[x_1^{(k)}; x_{z_k}^{(k)}]$ so that for $x_a \in [x_1^{(k)}; x_{z_k}^{(k)}]$ and $x_b \in [x_1^{(k)}; x_{z_k}^{(k)}]$

(59) if $x_a > x_b$, then $u(x_a) \leq u(x_b)$.

Since the coefficients of the linear function $\sum_{i=2}^{z_k-1} F_i^{W,(k)} \left(I_{i\text{-}1}^{p,(k)} - I_i^{p,(k)} \right)$ are entirely non-negative, then the minimum would be identified only for the smallest values of the unknown variables that obey the linear constraints, i.e.

(60) $F_k^{W,(k)} = F_k^{d,(k)}$, i=2, 3, ..., z_k–1.

So, in the special case of decreasing mean utilities the General Problem in 3.2.2 can be solved using the Wald criterion utilising the following simplification of the Universal Algorithm:

Simplified Algorithm 3

1) Calculate the mean utilities $I_i^{p,(k)}$ for i=1, 2,…, z_k-1 using (48) by numerically integrating the Riemann integral in the upper line.

2) Confirm the condition (58) and use (60) to find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly interpolated classical CDF I_k^{pW}.

3) Calculate the Wald-expected utility of the 1D p-fuzzy rational GL-I using (52).

3.2.3.4. Partially Linear-Interpolated Monotonic Utility under Wald criterion (Q=W)

Let the utility (19) be monotonic and partially linear-interpolated on the nodes (18). Then either condition (56) or condition (58) will hold. Because of that mean utilities are going to be either increasing or decreasing and the unknown inner quantile indices could be calculated respectively with (57) or with (60). The mean utilities do not need to be calculated. The *W*-expected utility can be calculated as in section 3.1.1.

So, in the special case of partially linear-interpolated monotonic utility the General Problem in 3.2.2 can be solved using the Wald criterion utilizing the following simplification of the Universal Algorithm:

Simplified Algorithm 4

1) If condition (56) holds then use (57) to find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly interpolated classical CDF $F_k^{pW}(.)$.

2) If condition (59) holds then use (60) to find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly interpolated classical CDF $F_k^{pW}(.)$.

3) Create $z^{(k)g}$ triplets (20) by merging the utility nodes (18) and the nodes $\{(x_l^{(k)}; F_i^{W,(k)})|k=1,2,...,z_k\}$, of the *p*-Wald approximated CDF: $F_k^{pW}(.)$.

4) Calculate the Wald-expected utility $E_k^{pW}(u| F_k^{pR})$ of the 1D p-fuzzy rational GL-I using the RHS of (21).

3.2.3.5. Increasing Arctan Utility under Wald Criterion (Q=W)

Let the utility be increasing and arctan approximated (22). Then condition (56) holds and therefore (55) holds. Because the mean utilities are increasing, the unknown inner quantile indices could be calculated with (57). The mean utilities do not need to be calculated. The W-expected utility can be calculated with modification of formula (29) where the nodes of the CDF $F_k^{(k)}$ have to be substituted with the nodes of the CDF $F_k^{W,(k)}$. The constants *A* and *B* have the meaning of A_{incr} and B_{incr} from (25).

(61) $$E_k^{pW}(u \mid F_k^{pR}) =$$
$$= A_{incr} \sum_{\substack{i=1 \\ x_i^{(k)} < x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{W,(k)} - F_i^{W,(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} C_i^{(k)} +$$
$$+ A_{incr} \sum_{\substack{i=1 \\ x_i^{(k)} = x_{i+1}^{(k)}}}^{z_k-1} (F_{i+1}^{W,(k)} - F_i^{W,(k)}) \arctan(a x_i^{(k)} - a x_0) + B_{incr}$$

Here $C_i^{(k)}$ are given in (30).

So, in the special case of increasing arctan-approximated utility the General Problem in 3.2.2 can be solved using the Wald criterion utilising the following simplification of the Universal Algorithm:

Simplified Algorithm 5

1) Use (57) to find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly interpolated classical CDF $F_k^{pW}(.)$.

2) Use (30) to calculate $C_i^{(k)}$ for i=1, 2,…, z_k-1

3) Use (25) to calculate the constants A_{incr} and B_{incr}.

4) Calculate the Wald-expected utility $E_k^{pW}(u| F_k^{pR})$ of the 1D p-fuzzy rational GL-I using (61).

3.2.3.6. Decreasing Arctan Utility under Wald Criterion (Q=W)

Let the utility be decreasing and arctan approximated (23). Then condition (59) holds and therefore (58) holds. Because the mean utilities are decreasing, the unknown inner quantile indices could be calculated with (60). The mean utilities do not need to be calculated. The W-expected utility can be calculated with modification of formula (29) where the nodes of the CDF $F_k^{(k)}$ have to be substituted with the nodes of the CDF $F_k^{W,(k)}$. The constants A and B have the meaning of A_{decr} and B_{decr} from (27).

(62) $$E_k^{pW}(u \mid F_k^{pR}) =$$
$$= A_{decr} \sum_{\substack{i=1 \\ x_i^{(k)} < x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{W,(k)} - F_i^{W,(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} C_i^{(k)} +$$
$$+ A_{decr} \sum_{\substack{i=1 \\ x_i^{(k)} = x_{i+1}^{(k)}}}^{z_k-1} (F_{i+1}^{W,(k)} - F_i^{W,(k)}) \arctan(a x_i^{(k)} - a x_0) + B_{decr}$$

Here, $C_i^{(k)}$ are given in (30).

So, in the special case of decreasing arctan-approximated utility the General Problem in 3.2.2 can be solved using the Wald criterion utilizing the following simplification of the Universal Algorithm:

Simplified Algorithm 6

1) Use (60) to find the inner quantile indices $F_i^{W,(k)}$, i=2, 3, ..., z_k–1 of the piece-wise partially linearly interpolated classical CDF $F_k^{pW}(.)$.

2) Use (30) to calculate $C_i^{(k)}$ for i=1, 2,…, z_k-1.

3) Use (27) to calculate the constants A_{decr} and B_{decr}.

4) Calculate the Wald-expected utility E_k^{pW} $(u| F_k^{pR})$ of the 1D p-fuzzy rational GL-I using (62).

3.2.4. Solution of the General Problem Using the Maximax Criterion (Q=¬W)

The maximax decision criterion under strict uncertainty assumes that the best outcome always occurs [Hackett, Luffrum, 1999]. The application of that concept in the case of a one-dimensional p-fuzzy rational GL-I implies to choose the quantile indices $F_i^{\neg W,(k)}$, i=2, 3, ..., z_k–1, so that to maximize the *pW*-expected utility of the lottery given in (40).

The rationale behind the maximax criterion is opposite to that of the Wald criterion. The required quantile indices $F_i^{\neg W,(k)}$, i=2, 3, ..., z_k–1, may be identified by trivial modifications of the proposed methods in section 3.2.3. The maximax-expected utility $E_k^{p\neg W}$ $(u| F_k^{pR})$ of the 1D p-fuzzy rational GL-I can be calculated using trivial adaptations of the algorithms proposed in section 3.2.3.

3.2.5. Solution of the General Problem Using the Hurwicz$_\alpha$ Criterion (Q= H$_\alpha$)

The Hurwicz$_\alpha$ decision criterion under strict uncertainty assumes that the choice of an alternative should be guided by a numerical index that is a weighted sum of the worst and the best outcome one can get from that alternative [Yager, 2006]. The application of that concept in the case of a one-dimensional p-fuzzy rational GL-I implies to choose the quantile indices $F_i^{H_\alpha,(k)}$, i=2, 3, ..., z_k–1 as weighted average of the quantile indices $F_i^{W,(k)}$ and $F_i^{\neg W(k)}$ from sections 3.2.3 and 3.2.4:

(63) $\quad F_i^{H_\alpha,(k)}, = \alpha F_i^{W,(k)}+(1\text{-}\alpha) F_i^{\neg W(k)}$, i=2, 3, ..., z_k–1.

The coefficient α ϵ [0;1] is a pessimism index that measured the pessimism of the DM.

The Hurwicz$_\alpha$-expected utility $E_k^{pH_\alpha}$ $(u| F_k^{pR})$ of the 1D p-fuzzy rational GL-I can be calculated again using trivial adaptations of the algorithms proposed in section 3.2.3.

4. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate how we apply the algorithms from section 3. These examples are part of previous larger studies (which are not the scope of the paper, hence will not be discussed in detail).

4.1. Example 1

A) Setup

Let's analyze the 1-D random variable *X*, which is the body temperature of a grown up human being in degrees Celsius, where X takes values in the interval [30; 42]. The preferences of the DM over the set of prizes are non-monotonic. Let us discuss the fifth alternative (*k=5*) in the set of lotteries L. Let a p-ribbon CDF $F_5^{pR}(.)$ is constructed over the values of *X* based on seven elicited inner quantile indices. Here, $z_5=9$ and the nodes are: ($x_1^{(5)}$ =30; $F_1^{d,(5)}$= $F_1^{u,(5)}$=0), ($x_2^{(5)}$ =31.5; $F_2^{d,(5)}$=0.07, $F_2^{u,(5)}$=0.13), ($x_3^{(5)}$ =33; $F_3^{d,(5)}$=0.23, $F_3^{u,(5)}$=0.31), ($x_4^{(5)}$ =34.5; $F_4^{d,(5)}$=0.41, $F_4^{u,(5)}$=0.51), ($x_5^{(5)}$ =36; $F_5^{d,(5)}$=0.51, $F_5^{u,(5)}$=0.65), ($x_6^{(5)}$ =37.5; $F_6^{d,(5)}$=0.68, $F_6^{u,(5)}$=0.80), ($x_7^{(5)}$ =39; $F_7^{d,(5)}$=0.76, $F_7^{u,(5)}$=0.88), ($x_8^{(5)}$ =40.5; $F_8^{d,(5)}$=0.85, $F_8^{u,(5)}$=0.95), ($x_9^{(5)}$ =42; $F_9^{d,(5)}$= $F_9^{u,(5)}$=1) The p-ribbon $F_5^{pR}(.)$ defines the p-fuzzy-rational GL-I – g_5^{pfr} =< $F_5^{pR}(x)$; x >.

The non-monotonic utility function of the DM in the interval [30; 42] is partially linearly interpolated on z^u=13 utility quantile indices: $u_1(30)=0$, $u_2(31)=0.06$, $u_3(32)=0.09$, $u_4(33)=0.15$, $u_5(34)=0.3$, $u_6(35)=0.55$, $u_7(36)=0.7$, $u_8(37)=0.6$, $u_9(38)=0.4$, $u_{10}(39)=0.2$, $u_{11}(40)=0.15$, $u_{12}(41)=0.1$, $u_{13}(42)=0$. The utility function id depicted on the lower section of *figure 1*.

The task is to approximate $F_5^{pR}(.)$ using Wald, maximax and Hurwicz$_{0.7}$ criteria and then calculate Wald, maximax and Hurwicz$_{0.7}$ expected utility of g_5^{pfr} .

Because the utility is partially linearly interpolated, we will apply simplified Algorithm 1 and two trivial modifications of it. As long as the first two steps do not depend on the approximation of the CDF, the mean utilities $I_i^{p,(5)}$, for i=1,2, …, 8 can be calculated for all three tasks. For this purpose, we need to use approximation of utility of type (20), neglecting the $F_i^{(5)g}(.)$. Here, $z^{(5)g}$ is 17 and the triplets (20) convert into pairs, as follows:

($x_1^{(5)g}$=30; $u_i^{(5)g}$=0), ($x_2^{(5)g}$=31; $u_2^{(5)g}$=0.06), ($x_3^{(5)g}$=31.5; $u_3^{(5)g}$=0.075), ($x_4^{(5)g}$=32; $u_4^{(5)g}$=0.09), ($x_5^{(5)g}$=33; $u_5^{(5)g}$=0.15), ($x_6^{(5)g}$=34; $u_6^{(5)g}$=0.30), ($x_7^{(5)g}$=34.5; $u_7^{(5)g}$=0.425), ($x_8^{(5)g}$=35; $u_8^{(5)g}$=0.55), ($x_9^{(5)g}$=36; $u_9^{(5)g}$=0.70), ($x_{10}^{(5)g}$=37; $u_{10}^{(5)g}$=0.60), ($x_{11}^{(5)g}$=37.5; $u_{11}^{(5)g}$=0.50), ($x_{12}^{(5)g}$=38; $u_{12}^{(5)g}$=0.40), ($x_{13}^{(5)g}$=39; $u_{13}^{(5)g}$=0.20), ($x_{14}^{(5)g}$=40; $u_{14}^{(5)g}$=0.15), ($x_{15}^{(5)g}$=40.5; $u_{15}^{(5)g}$=0.125), ($x_{16}^{(5)g}$=41; $u_{16}^{(5)g}$=0.10), ($x_{17}^{(5)g}$=42; $u_{17}^{(5)g}$=0)

The mean utilities must be calculated using (53) and (54). For example, let us find $I_3^{p,(5)}$. Using (53), i_{beg}=5, because $x_5^{(5)g}$=$x_3^{(5)}$=33, and i_{end}=7, because $x_7^{(5)g}$=$x_4^{(5)g}$=34.5. Then the mean utility $I_3^{p,(5)}$ may be calculated using the first line of (54), because $x_4^{(5)}$> $x_3^{(5)}$: $I_3^{p,(5)}=\frac{1}{2(34.5\text{-}33)}$[(34-33) (0.3+0.15)+(34.5-34)(0.425+0.3)].

In the same way, we can find all mean utilities: $I_1^{p,(5)}$=0.0425, $I_2^{p,(5)}$ =0.1075, $I_3^{p,(5)}$ =0.2708, $I_4^{p,(5)}$ =0.5792, $I_5^{p,(5)}$ =0.6167, $I_6^{p,(5)}$ =0.35, $I_7^{p,(5)}$ =0.1625, $I_8^{p,(5)}$ =0.0708. Note that since the abscissas are different, we will always **use the upper line of (54) for the mean utilities.**

B) Calculation of Wald expected utility

According to step 3 of Simplified Algorithm 1 the inner quantiles $F_i^{W,(5)}$ for i=2, 3,…,8 can be found by solving the following task of linear programming: minimize $-0.065F_2^{W,(5)}$ $-0.1633F_3^{W,(5)}$ $-0.3083F_4^{W,(5)}$ $-0.0375F_5^{W,(5)}$ $+0.2667F_6^{W,(5)}$ $+0.1875F_7^{W,(5)}$ $+0.0917F_8^{W,(5)}$ under the following 3_{z5}–5=22 linear constraints: $0\le F_2^{W,(5)}\le F_3^{W,(5)}\le\ldots\le F_8^{W,(5)}\le 1$, $0.07\le F_2^{W,(5)}\le 0.13$, $0.23\le F_3^{W,(5)}\le 0.31$, $0.41\le F_4^{W,(5)}\le 0.51$, $0.51\le F_5^{W,(5)}\le 0.65$, $0.68\le F_6^{W,(5)}\le 0.80$, $0.76\le F_7^{W,(5)}\le 0.88$, $0.85\le F_8^{W,(5)}\le 0.95$. This task was solved using the linprog function in MATLAB with the following results: $F_2^{W,(5)}=0.13$, $F_3^{W,(5)}=0.31$, $F_4^{W,(5)}=0.51$, $F_5^{W,(5)}=0.65$, $F_6^{W,(5)}=0.68$, $F_7^{W,(5)}=0.76$, $F_8^{W,(5)}=0.85$.

So $F_5^{pR}(.)$ is approximated with $F_5^{pW}(.)$ on the nodes ($x_1^{(5)}=30$; $F_1^{W,(5)}=0$), ($x_2^{(5)}=31.5$; $F_2^{W,(5)}=0.13$), ($x_3^{(5)}=33$; $F_3^{W,(5)}=0.31$), ($x_4^{(5)}=34.5$; $F_4^{W,(5)}=0.51$), ($x_5^{(5)}=36$; $F_5^{W,(5)}=0.65$), ($x_6^{(5)}=37.5$; $F_6^{W,(5)}=0.68$), ($x_7^{(5)}=39$; $F_7^{W,(5)}=0.76$), ($x_8^{(5)}=40.5$; $F_8^{W,(5)}=0.85$), ($x_9^{(5)}=42$; $F_9^{W,(5)}=1$). The graphics of the $F_5^{pW}(.)$ and its density are given on *figure 1*. Then g_5^{pfr} is approximated using the 1-D pW-GL-I – $g_5^{pW}=<F_5^{pW}(x); x>$. Using (52), the expected utility of the pW-GL-I is $E_5^{pW}(u|F_5^{pW})=0.2319$, which is the p-Wald expected utility of g_5^{pfr} (see *figure 1*).

C) Calculation of maximax expected utility

Using trivial modification of the Simplified Algorithm 1, the inner quantiles $F_i^{\neg W,(5)}$ for i=2, 3, …, 8 can be found by solving the following task of linear programming:
minimize

$$\begin{pmatrix} -0.065F^{\neg W,(5)}-0.1633F^{\neg W,(5)}-0.3083F^{\neg W,(5)} \\ -0.0375F_5^{\neg W,(5)}+0.2667F_6^{\neg W,(5)}+0.1875F_7^{\neg W,(5)} \\ +0.0917F_8^{\neg W,(5)} \end{pmatrix}$$

under the following $3z_5$–5=22 linear constraints: $0\le F_2^{\neg W,(5)}\le F_3^{\neg W,(5)}\le\ldots\le F_8^{\neg W,(5)}\le 1$, $0.07\le F_2^{\neg W,(5)}\le 0.13$, $0.23\le F_3^{\neg W,(5)}\le 0.31$, $0.41\le F_4^{\neg W,(5)}\le 0.51$, $0.51\le F_5^{\neg W,(5)}\le 0.65$, $0.68\le F_6^{\neg W,(5)}\le 0.80$, $0.76\le F_7^{\neg W,(5)}\le 0.88$, $0.85\le F_8^{\neg W,(5)}\le 0.95$. This task was solved using the linprog function in MATLAB with the following results: $F_2^{\neg W,(5)}=0.07$, $F_3^{\neg W,(5)}=0.23$, $F_4^{\neg W,(5)}=0.41$, $F_5^{\neg W,(5)}=0.51$, $F_6^{\neg W,(5)}=0.80$, $F_7^{\neg W,(5)}=0.88$, $F_8^{\neg W,(5)}=0.95$. So $F_5^{pR}(.)$ is approximated with $F_5^{p\neg W}(.)$ on the nodes ($x_1^{(5)}=30$; $F_1^{\neg W,(5)}=0$), ($x_2^{(5)}=31.5$; $F_2^{\neg W,(5)}=0.07$), ($x_3^{(5)}=33$; $F_3^{\neg W,(5)}=0.23$), ($x_4^{(5)}=34.5$; $F_4^{\neg W,(5)}=0.41$), ($x_5^{(5)}=36$; $F_5^{\neg W,(5)}=0.51$), ($x_6^{(5)}=37.5$; $F_6^{\neg W,(5)}=0.80$), ($x_7^{(5)}=39$; $F_7^{\neg W,(5)}=0.88$), ($x_8^{(5)}=40.5$; $F_8^{\neg W,(5)}=0.95$), ($x_9^{(5)}=42$; $F_9^{\neg W,(5)}=1$). The graphics of the $F_5^{pW}(.)$ and its density are given on *figure 2*. The expected utility of the p¬W-GL-I may be calculated using a trivial modification of (52):

$$E_k^{p\neg W}(u\,|\,F_k^{pR})=I_{z_k-1}^{p,(k)}+\sum_{i=2}^{z_k-1}F_i^{\neg W,(k)}\left(I_{i-1}^{p,(k)}-I_i^{p,(k)}\right)$$

Using the above formula, we find the expected utility of the p¬W-GL-I to be $E_5^{p\neg W}(u|F_5^{p\neg W})=0.3486$, which is the maximax expected utility of the g_5^{pfr} (see *figure 2*)

D) Calculation of Hurwicz$_\alpha$ expected utility

Here, the pessimism index α=0.7. Using trivial modification of the Simplified Algorithm 1, the inner quantile indices can be calculated using (63): $F_2^{H_{0.7}}=0.7F_2^{W,(5)}+(1\text{-}0.7)F_2^{\neg W(5)}=$ $=0.7\times0.13+0.3\times0.07=0.112$, $F_3^{H_{0.7}}=0.286$, $F_4^{H_{0.7}}=0.48$, $F_5^{H_{0.7}}=0.608$, $F_6^{H_{0.7}}=0.716$, $F_7^{H_{0.7}}=0.796$, $F_8^{H_{0.7}}=0.88$. Then $F_5^{pR}(.)$ is approximated by $F_5^{pH_{0.7}}(.)$ on the nodes ($x_1^{(5)}=30$; $F_1^{H_{0.7},(5)}=0$), ($x_2^{(5)}=31.5$; $F_2^{H_{0.7},(5)}=0.112$), ($x_3^{(5)}=33$; $F_3^{H_{0.7},(5)}=0.286$), ($x_4^{(5)}=34.5$; $F_4^{H_{0.7},(5)}=0.48$), ($x_5^{(5)}=36$; $F_5^{H_{0.7},(5)}=0.608$), ($x_6^{(5)}=37.5$; $F_6^{H_{0.7},(5)}=0.716$), ($x_7^{(5)}=39$; $F_7^{H_{0.7},(5)}=0.796$), ($x_8^{(5)}=40.5$; $F_8^{H_{0.7},(5)}=0.88$), ($x_9^{(5)}=42$; $F_9^{H_{0.7},(5)}=1$). The graphics of the $F_5^{pH_{0.7}}(.)$ and its density are given on *figure 3*. Then g_5^{pfr} is approximated using the one-dimensional p$H_{0.7}$-GL-I – $g_5^{pH_{0.7}}=<F_5^{pH_{0.7}}(.); x>$. The expected utility of the p $H_{0.7}$-GL-I may be calculated using another trivial modification of (52):

$$E_k^{pH_{0.7}}(u\,|\,F_k^{pR})=I_{z_k-1}^{p,(k)}+\sum_{i=2}^{z_k-1}F_i^{H_{0.7},(k)}\left(I_{i-1}^{p,(k)}-I_i^{p,(k)}\right)$$

Using the above formula, we find the expected utility of the p $H_{0.7}$-GL-I to be $E_5^{pH_{0.7}}(u|F_5^{pH_{0.7}})=0.2669$, which is the Hurwicz$_{0.7}$ expected utility of the g_5^{pfr} (see *figure 3*).

4.2. Example 2

A) Setup

Let's analyze the 1-D random variable X, which is the investment in USD in a home heating system, where *X* takes values in the interval [4650; 5150]. The preferences of the DM over the set of prizes are monotonically decreasing. Let us discuss the seventh alternative (*k*=7) in the set of lotteries L. Let a *p*-ribbon CDF $F_7^{pR}(.)$ is constructed over the values of X based on nine elicited inner quantile indices: ($x_1^{(7)}=4650$; $F_1^{d,(7)}=F_1^{u,(7)}=0$), ($x_2^{(7)}=4700$; $F_2^{d,(7)}=0.11$, $F_2^{u,(7)}=0.15$), ($x_3^{(7)}=4750$; $F_3^{d,(7)}=0.37$, $F_3^{u,(7)}=0.44$), ($x_4^{(7)}=4800$; $F_4^{d,(7)}=0.52$, $F_4^{u,(7)}=0.62$), ($x_5^{(7)}=4850$; $F_5^{d,(7)}=0.63$, $F_5^{u,(7)}=0.75$), ($x_6^{(7)}=4900$; $F_6^{d,(7)}=0.73$, $F_6^{u,(7)}=0.87$), ($x_7^{(7)}=4950$; $F_7^{d,(7)}=0.85$, $F_7^{u,(7)}=0.95$), ($x_8^{(7)}=5000$; $F_8^{d,(7)}=0.93$, $F_8^{u,(7)}=0.97$), ($x_9^{(7)}=5050$; $F_9^{d,(7)}=0.96$, $F_9^{u,(7)}=0.98$), ($x_{10}^{(7)}=5100$; $F_{10}^{d,(7)}=0.97$, $F_{10}^{u,(7)}=0.99$), ($x_{11}^{(7)}=5150$; $F_{11}^{d,(7)}=F_{11}^{u,(7)}=1$). The *p*-ribbon $F_7^{pR}(.)$ defines the p-fuzzy-rational GL-I – $g_7^{pfr}=<F_7^{pR}(x); x>$.

The monotonically decreasing utility function of the DM in the interval [0; 9000] is constructed on five inner utility quantile indices: $u_2=0.8$, $u_3=0.65$, $u_4=0.5$, $u_5=0.35$, $u_6=0.20$ and their utility quantiles were elicited as follows:

$\hat{x}_{u_2}\in\left[\hat{x}_{u_2}^d;\hat{x}_{u_2}^u\right]\equiv[2000;\ 2600]$, $\hat{x}_{u_3}\in\left[\hat{x}_{u_3}^d;\hat{x}_{u_3}^u\right]\equiv[3000;\ 3600]$, $\hat{x}_{u_4}\in\left[\hat{x}_{u_4}^d;\hat{x}_{u_4}^u\right]\equiv[3700;\ 4500]$, $\hat{x}_{u_5}\in\left[\hat{x}_{u_5}^d;\hat{x}_{u_5}^u\right]\equiv[4700;\ 5300]$, $\hat{x}_{u_6}\in\left[\hat{x}_{u_6}^d;\hat{x}_{u_6}^u\right]\equiv[5800;\ 6200]$.

The utility function is arctan approximated on the elicited nodes using (23), as demonstrated in [Nikolova et al., 2018]. The optimal parameters of the analytical utility function are $x_d=0$, $x_u=9000$, $x_0=3982$, a=4e–4. The resulting utility function is:

$$u(x)=\frac{\arctan(ax_u-ax_0)-\arctan(ax-ax_0)}{\arctan(ax_u-ax_0)-\arctan(ax_d-ax_0)}=$$

$$=\frac{\arctan(2.0072)-\arctan\left(4\times10^{-4}\times x-1.5928\right)}{\arctan(2.0072)-\arctan(-1.5928)}$$

The approximated utility function and its corresponding local risk aversion function (in this case interpret as local risk proneness) are given in *figure 4*. It shows that the approximation is acceptable because the utility function passes through the uncertainty interval of each of the nodes.

The task is to approximate $F_7^{pR}(.)$ using Wald, maximax and Hurwicz$_{0.7}$ criteria and then calculate Wald, maximax and Hurwicz$_{0.7}$ expected utility of g_7^{pfr}.

Because we have decreasing arctan utility, the three problems stated will be solved using the simplified Algorithm 6 and two trivial modifications of it. As long as the second and third step do not depend on the approximation of the CDF, then the constants $C_i^{(7)}$, for i=1,2, …, 10, A_{decr} and B_{decr} can be calculated for all three tasks: $C_1^{(7)}$=13.5201, $C_2^{(7)}$=14.4439, $C_3^{(7)}$=15.3576, $C_4^{(7)}$=16.2609, $C_5^{(7)}$=17.1533, $C_6^{(7)}$=18.0344, $C_7^{(7)}$=18.9040, $C_8^{(7)}$=19.7618, $C_9^{(7)}$=20.6074, $C_{10}^{(7)}$=21.4407, A_{decr}=-0.4720, B_{decr}=0.5232.

B) Calculation of Wald expected utility

According to step 1 of simplified algorithm 6, the inner quantile indices are set to their lower bounds using (60): $F_2^{W,(7)}$=0.11, $F_3^{W,(7)}$=0.37, $F_4^{W,(7)}$=0.52, $F_5^{W,(7)}$=0.63, $F_6^{W,(7)}$=0.73, $F_7^{W,(7)}$ =0.85, $F_8^{W,(7)}$=0.93, $F_9^{W,(7)}$=0.96, $F_{10}^{W,(7)}$=0.97. Then $F_7^{pR}(.)$ is approximated by $F_7^{pW}(.)$ on the nodes ($x_1^{(7)}$=4650; $F_1^{W,(7)}$=0), ($x_2^{(7)}$=4700; $F_2^{W,(7)}$=0.11), ($x_3^{(7)}$=4750; $F_3^{W,(7)}$=0.37), ($x_4^{(7)}$=4800; $F_4^{W,(7)}$=0.52), ($x_5^{(7)}$=4850; $F_5^{W,(7)}$=0.63), ($x_6^{(7)}$=4900; $F_6^{W,(7)}$=0.73), ($x_7^{(7)}$=4950; $F_7^{W,(7)}$=0.85), ($x_8^{(7)}$=5000; $F_8^{W,(7)}$=0.93), ($x_9^{(7)}$=5050; $F_9^{W,(7)}$=0.96), ($x_{10}^{(7)}$=5100; $F_{10}^{W,(7)}$=0.97), ($x_{11}^{(7)}$=5150; $F_{11}^{W,(7)}$=1). The graphics of the $F_7^{pW}(x)$ and its density are given on *figure 5*. Then g_7^{pfr} is approximated using the one-dimensional pW-GL-I – g_7^{pW}=<$F_7^{pW}(x)$; x>. Using (62), the expected utility of the pW-GL-I is $E_7^{pW}(u|F_7^{pW})$=0.3704, which is the Wald-expected utility of the g_7^{pfr} (see *figure 5*). Note that using (62), the second member of the equation is skipped because the quantiles are different.

C) Calculation of maximax expected utility

Using trivial modification of the Simplified Algorithm 6, the inner quantile indices are set to their upper bounds: $F_2^{\neg W,(7)}$ =0.15, $F_3^{\neg W,(7)}$=0.44, $F_4^{\neg W,(7)}$=0.62, $F_5^{\neg W,(7)}$=0.75, $F_6^{\neg W,(7)}$ =0.87, $F_7^{\neg W,(7)}$=0.95, $F_8^{\neg W,(7)}$=0.97, $F_9^{\neg W,(7)}$=0.98, $F_{10}^{\neg W,(7)}$=0.99. Then $F_7^{pR}(.)$ is approximated by $F_7^{p\neg W}(.)$ on the nodes ($x_1^{(7)}$=4650; $F_1^{\neg W,(7)}$=0), ($x_2^{(7)}$=4700; $F_2^{\neg W,(7)}$=0.15), ($x_3^{(7)}$=4750; $F_3^{\neg W,(7)}$=0.44), ($x_4^{(7)}$=4800; $F_4^{\neg W,(7)}$=0.62), ($x_5^{(7)}$=4850; $F_5^{\neg W,(7)}$=0.75), ($x_6^{(7)}$=4900; $F_6^{\neg W,(7)}$=0.87), ($x_7^{(7)}$=4950; $F_7^{\neg W,(7)}$=0.95), ($x_8^{(7)}$=5000; $F_8^{\neg W,(7)}$=0.97), ($x_9^{(7)}$=5050; $F_9^{\neg W,(7)}$=0.98), ($x_{10}^{(7)}$=5100; $F_{10}^{\neg W,(7)}$=0.99), ($x_{11}^{(7)}$=5150; $F_{11}^{\neg W,(7)}$=1). The graphics of the $F_7^{p\neg W}(.)$ and its density are given on *figure 6*. Then g_7^{pfr} is approximated using the one-dimensional p¬W-GL-I – $g_7^{p\neg W}$=<$F_7^{p\neg W}(.)$;x>. The expected utility of the p -GL-I may be calculated using a trivial modification of (62):

$$E_k^{p\neg W}(u \mid F_k^{pR}) = A_{decr} \sum_{\substack{i=1 \\ x_i^{(k)} < x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{\neg W,(k)} - F_i^{\neg W,(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} C_i^{(k)} +$$

$$+ A_{decr} \sum_{\substack{i=1 \\ x_i^{(k)} = x_{i+1}^{(k)}}}^{z_k-1} (F_{i+1}^{\neg W,(k)} - F_i^{\neg W,(k)}) \arctan(a x_i^{(k)} - a x_0) + B_{decr}$$

Using the above formula, we find the expected utility of the p¬*W*-GL-I to be $E_7^{p\neg W}(u|F_7^{p\neg W})$=0.3758, which is the maximax expected utility of the g_7^{pfr} (see *figure 6*). Note that in the above formula, the second member of the equation is skipped because the quantiles are different.

D) Calculation of Hurwicz$_{0.7}$ expected utility

Here, the pessimism index α=0.7. Using trivial modification of the Simplified Algorithm 6, the inner quantile indices can be calculated using (63): $F_2^{H0.7,(7)}=0.7F_2^{W,(7)}+(1\text{-}0.7)F_2^{\neg W,(7)}$ =0.7×0.11+0.3×0.15=0.122, $F_3^{H0.7,(7)}$=0.391, $F_4^{H0.7,(7)}$=0.55, $F_5^{0.7,(7)}$=0.666, $F_6^{H0.7,(7)}$=0.786, $F_7^{H0.7,(7)}$=0.88, $F_8^{H0.7,(7)}$=0.942, $F_9^{H0.7,(7)}$=0.966, $F_{10}^{H0.7,(7)}$=0.976. Then $F_7^{pR}(.)$ is approximated by $F_7^{pH0.7}(.)$ on the nodes ($x_1^{(7)}$=4650; $F_1^{H0.7,(7)}$=0), ($x_2^{(7)}$=4700; $F_2^{H0.7,(7)}$ =0.122), ($x_3^{(7)}$=4750; $F_3^{H0.7,(7)}$=0.391), ($x_4^{(7)}$=4800; $F_4^{H0.7,(7)}$ =0.55), ($x_5^{(7)}$=4850; $F_5^{H0.7,(7)}$=0.666), ($x_6^{(7)}$=4900; $F_6^{H0.7,(7)}$ =0.786), ($x_7^{(7)}$=4950; $F_7^{H0.7,(7)}$=0.88), ($x_8^{(7)}$=5000; $F_8^{H0.7,(7)}$=0.942), ($x_9^{(7)}$=5050; $F_9^{H0.7,(7)}$=0.966), ($x_{10}^{(7)}$=5100; $F_{10}^{H0.7,(7)}$ =0.976), ($x_{11}^{(7)}$=5150; $F_{11}^{H0.7,(7)}$=1). The graphics of the $F_7^{pH0.7}(.)$ and its density are given on *figure 7*. Then is approximated using the one-dimensional pH$_{0.7}$-GL-I – $g_7^{pH0.7}$=<$F_7^{pH0.7}(.)$; x>. The expected utility of the p H$_{0.7}$-GL-I may be calculated using another trivial modification of (62):

$$E_k^{pH_{0.7}}(u \mid F_k^{pR}) = A_{decr} \sum_{\substack{i=1 \\ x_i^{(k)} < x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{H_{0.7},(k)} - F_i^{H_{0.7},(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} C_i^{(k)} +$$

$$+ A_{decr} \sum_{\substack{i=1 \\ x_i^{(k)} = x_{i+1}^{(k)}}}^{z_k-1} (F_{i+1}^{H_{0.7},(k)} - F_i^{H_{0.7},(k)}) \arctan(a x_i^{(k)} - a x_0) + B_{decr}$$

Using the above formula, we find the expected utility of the p H$_{0.7}$-GL-I to be $E_7^{pH0.7}(u|F_7^{pH0.7})$ =0.3722, which is the Hurwicz$_{0.7}$ expected utility of the g_7^{pfr} (see *figure 7*). Note that in the above formula, the second member of the equation is skipped because the quantiles are different.

5. Conclusion

We discussed alternatives under risk modelled as GL-I, where uncertainty was modeled using p-ribbon distributions (i.e. constructed on interval estimates of quantile indices of a fuzzy-rational DM). We developed procedures to approximate such ribbon distributions with classical CDFs using three criteria under strict uncertainty. Subsequently, the procedures also demonstrated how to rank the p-fuzzy-rational GL-Is according to the Wald, maximax and Hurwicz$_\alpha$ criteria. Those procedures also adapted to the type of preferences of the DM (monotonically increasing/decreasing or non-monotonic) and the way the utility function was constructed (using partial linear interpolation or arctan approximation). Then we provided simplified algorithms to rank p-fuzzy-rational GL-I using the Wald expected utility criterion and also derived the maximax and Hurwicz$_\alpha$ expected utility criteria using trivial modifications. The numerical examples illustrated cases of non-monotonic partially linearly interpolated utility and of decreasing arctan approximated utility functions. The choice of α=0.7 assumed a pessimistic DM and justified results like those of the Wald expected utility.

The Hurwicz$_\alpha$ approach to approximate the p-ribbon functions makes the probabilities dependent on the preferences of the DM. Although this is not compliant with the expected utility paradigm, the DM who uses the Hurwicz$_\alpha$ criterion is “probabilistically sophisticated non-expected utility maximizer” as suggested in [Machina, Schmeidler, 1992]. Similar ideas about preference-dependent probabilities is also suggested by [Augustin, 2001].

Our earlier work [Tenekedjiev, Nikolova, 2008] discussed the application of the same criteria to rank fuzzy-rational alternatives, but for the case of ordinary lotteries. As indicated there, the use of Wald, maximax and Hurwicz$_\alpha$ ¬wass justified as these methods (while not perfectly rational) are well approbated descriptive techniques whose properties are well known. It was also clarified that the standard approach to tasks where uncertainties are elicited in an interval form would be to replace the intervals by their mid points (following various approaches). In our task of GL-I$_9$ this would mean to linearly interpolate the CDFs on the midpoints of the elcitied intervals. The drawbacks of this approach outlined in [Tenekedjiev, Nikolova, 2008] will not appear for the case of continuous probability distributions. However, the main issue would remain that we would artificially replace the uncertainty interval by a point estimate. On one hand this would impose high precision of estimates, which is not representative of the opinion of a fuzzy-rational DM. On the other, we would deprive the analysis of the information that the uncertainty interval carries, as it demonstrates what the DM does or does not know. Hence a strong side of our approach is that we can solve tasks with partially quantified uncertainty while utilizing the full volume of information about the uncertainty. Our criteria can smoothly transition from tasks of strict uncertainty (i.e. when uncertainty intervals are very wide) to tasks under risk (i.e. when uncertainty intervals are rather tight). Another alternative solution would need to rely on the information-gap theory to solve tasks under severe uncertainty [Ben-Haim, 2001]. However, this traditionally produces higher ambiguity in the solution.

Another strong side of our study is that in addition to the standard elements in any decision-making situation (degree of belief, value system and risk attitude) we included the pessimism-optimism of the DM as a tool to approximate the p-ribbon functions and rank the *p*-fuzzy-rational GL-Is.

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Appendix – Proof of Formulae (29) and (30)

Let a partially linearly interpolated CDF is given in the form (17). Let the utility function is arctan-approximated and given in (28). Then the expected utility of the GL-I in (15) is

$$\text{(A1)}\quad E_k\left(u \mid F_k\right) = A \sum_{\substack{i=1 \\ x_i^{(k)} < x_{i+1}^{(k)}}}^{z_k - 1} \frac{F_{i+1}^{(k)} - F_i^{(k)}}{x_{i+1}^{(k)} - x_i^{(k)}} C_i^{(k)} + A \sum_{\substack{i=1 \\ x_i^{(k)} = x_{i+1}^{(k)}}}^{z_k - 1} (F_{i+1}^{(k)} - F_i^{(k)}) \arctan(a x_i^{(k)} - a x_0) + B$$

where

$$\text{(A2)}\quad C_i^{(k)} = (x_{i+1}^{(k)} - x_0)\arctan(a x_{i+1}^{(k)} - a x_0) - (x_i^{(k)} - x_0)\arctan(a x_i^{(k)} - a x_0) + \frac{1}{2a}\ln\frac{1 + (a x_i^{(k)} - a x_0)^2}{1 + (a x_{i+1}^{(k)} - a x_0)^2}$$

Proof

Let us solve the auxiliary definite integral $\int_{x_i^{(k)}}^{x_{i+1}^{(k)}} \arctan\left(ax - ax_0\right) \mathrm{d}x$ using integration by parts

$$C_i^{(k)} = \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} \arctan\left(ax - ax_0\right)\mathrm{d}x = \left[x \times \arctan\left(ax - ax_0\right)\right]_{x_i^{(k)}}^{x_{i+1}^{(k)}} - \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} x\, \mathrm{d}\left[\arctan\left(ax - ax_0\right)\right] =$$

$$= \left[x \times \arctan\left(ax - ax_0\right)\right]_{x_i^{(k)}}^{x_{i+1}^{(k)}} - \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} \frac{ax}{1 + \left(ax - ax_0\right)^2}\mathrm{d}x$$

Let us substitute in the integral $y = ax - ax_0$, which will produce $\mathrm{d}x = (1/a)\mathrm{d}y$. Then

$$C_i^{(k)} = \left[x \times \arctan\left(ax - ax_0\right)\right]_{x_i^{(k)}}^{x_{i+1}^{(k)}} - \int_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} \frac{a\left(y/a + x_0\right)}{1 + y^2}\frac{1}{a}\mathrm{d}y =$$

$$= \left[x \times \arctan\left(ax - ax_0\right)\right]_{x_i^{(k)}}^{x_{i+1}^{(k)}} - \int_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} \frac{\left(y/a + x_0\right)}{1 + y^2}\mathrm{d}y =$$

$$= \left[x \times \arctan\left(ax - ax_0\right)\right]_{x_i^{(k)}}^{x_{i+1}^{(k)}} - x_0 \int_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} \frac{1}{1 + y^2}\mathrm{d}y - \frac{1}{a}\int_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} \frac{y}{1 + y^2}\mathrm{d}y =$$

$$= \left[x \times \arctan\left(ax - ax_0\right)\right]_{x_i^{(k)}}^{x_{i+1}^{(k)}} - x_0\left[\arctan\left(y\right)\right]_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} - \frac{1}{2a}\int_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} \frac{1}{1 + y^2}\mathrm{d}\left(y^2 + 1\right) =$$

$$= \left[x \times \arctan\left(ax - ax_0\right)\right]_{x_i^{(k)}}^{x_{i+1}^{(k)}} - x_0\left[\arctan\left(y\right)\right]_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} - \frac{1}{2a}\left[\ln\left|y^2 + 1\right|\right]_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} =$$

$$= x_{i+1}^{(k)}\arctan\left(a x_{i+1}^{(k)} - a x_0\right) - x_i^{(k)}\arctan\left(a x_i^{(k)} - a x_0\right) - x_0\arctan\left(a x_{i+1}^{(k)} - a x_0\right) + x_0\arctan\left(a x_i^{(k)} - a x_0\right) - \frac{1}{2a}\left[\ln\left|y^2 + 1\right|\right]_{a x_i^{(k)} - a x_0}^{a x_{i+1}^{(k)} - a x_0} =$$

$$= \left(x_{i+1}^{(k)} - x_0\right)\arctan\left(a x_{i+1}^{(k)} - a x_0\right) - \left(x_i^{(k)} - x_0\right)\arctan\left(a x_i^{(k)} - a x_0\right) + \frac{1}{2a}\ln\frac{1 + \left(a x_i^{(k)} - a x_0\right)^2}{1 + \left(a x_{i+1}^{(k)} - a x_0\right)^2}$$

This proves (A2).

In the case of partially linearly interpolated CDF, the expected utility integral is given in (18) as shown in [Nikolova et al., 2010a]. We can plug the utility (28) into (18) and use (A2). So:

$$E_k\left(u \mid F_k\right)=\int_{-\infty}^{+\infty} u(x) dF_k(x)=\sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} u(x)\,\mathrm{d}x+\sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}(F_{i+1}^{(k)}-F_i^{(k)}) u\left(x_i^{(k)}\right)=$$

$$=\sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} \int_{x_i^{(k)}}^{x_{i+1}^{(k)}}\left[A \times \arctan\left(a x-a x_0\right)+B\right] \mathrm{d}x+\sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}(F_{i+1}^{(k)}-F_i^{(k)}) u\left(x_i^{(k)}\right)=$$

$$=\sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}}\left[A \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} \arctan\left(a x-a x_0\right) \mathrm{d}x+B \int_{x_i^{(k)}}^{x_{i+1}^{(k)}} \mathrm{d}x\right]+\sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}(F_{i+1}^{(k)}-F_i^{(k)}) u\left(x_i^{(k)}\right)=$$

$$=\sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}}\left[A C_i^{(k)}+B\left(x_{i+1}^{(k)}-x_i^{(k)}\right)\right]+\sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}(F_{i+1}^{(k)}-F_i^{(k)}) u\left(x_i^{(k)}\right)=$$

$$=A \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} C_i^{(k)}+B \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right)+\sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}(F_{i+1}^{(k)}-F_i^{(k)})\left[A \times \arctan\left(a x-a x_0\right)+B\right]=$$

$$=A \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} C_i^{(k)}+B \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right)+A \sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}(F_{i+1}^{(k)}-F_i^{(k)}) \arctan\left(a x-a x_0\right)+B \sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}(F_{i+1}^{(k)}-F_i^{(k)})=$$

$$=A \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} C_i^{(k)}+A \sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right) \arctan\left(a x-a x_0\right)+B \sum_{i=1}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right)=$$

$$=A \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} C_i^{(k)}+A \sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right) \arctan\left(a x-a x_0\right)+B\left(F_2^{(k)}-F_1^{(k)}+F_3^{(k)}-F_2^{(k)}+\ldots.+F_{z_k}^{(k)}-F_{z_k-1}^{(k)}\right)=$$

$$=A \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} C_i^{(k)}+A \sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right) \arctan\left(a x-a x_0\right)+B\left(-F_1^{(k)}+F_{z_k}^{(k)}\right)=$$

$$=A \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} C_i^{(k)}+A \sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right) \arctan\left(a x-a x_0\right)+B(-0+1)=$$

$$=A \sum_{\substack{i=1\\x_i^{(k)}<x_{i+1}^{(k)}}}^{z_k-1} \frac{F_{i+1}^{(k)}-F_i^{(k)}}{x_{i+1}^{(k)}-x_i^{(k)}} C_i^{(k)}+A \sum_{\substack{i=1\\x_i^{(k)}=x_{i+1}^{(k)}}}^{z_k-1}\left(F_{i+1}^{(k)}-F_i^{(k)}\right) \arctan\left(a x-a x_0\right)+B$$

This proves (A1).

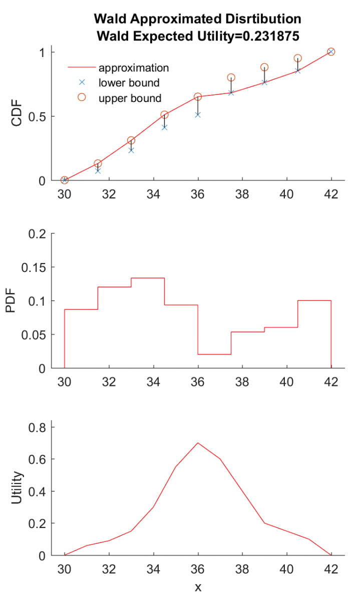

Figure 1. Graphics of $F_5^{pW}(x)$, density (PDF) and utility function of the DM over the values in X from Example 1 in the interval [30;42]

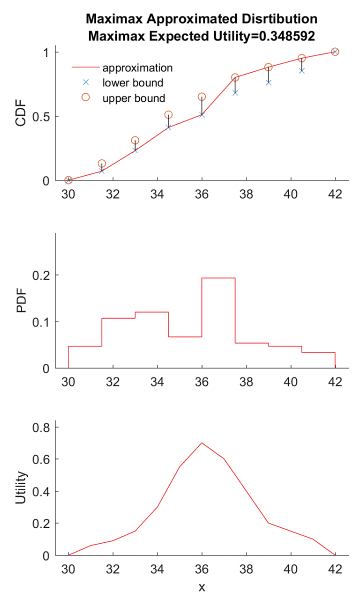

Figure 2. Graphics of $F_5^{p\neg W}(x)$, density (PDF) and utility function of the DM over the values in X from Example 1 in the interval [30;42]

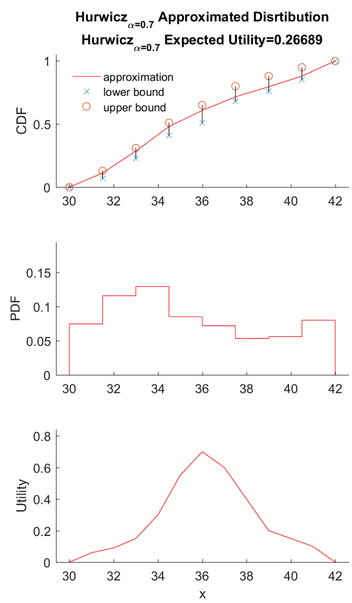

Figure 3. Graphics of $F_5^{pH_{0.7}}(x)$, density (PDF) and utility function of the DM over the values in X from Example 1 in the interval [30;42]

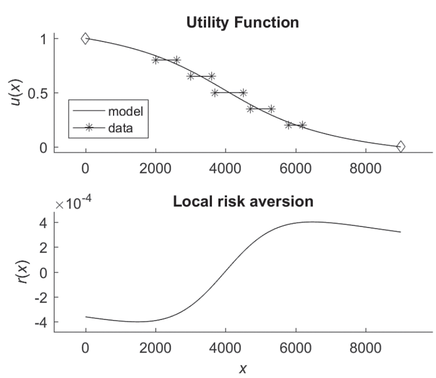

Figure 4. Approximated utility function and its corresponding local risk aversion function (in this case interpret as local risk proneness) for Example 2

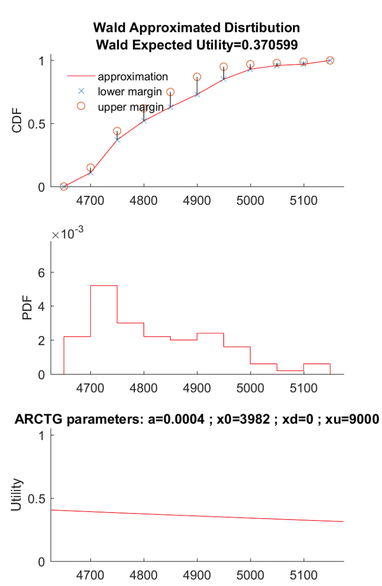

Figure 5. Graphics of $F_7^{pW}(x)$, density (PDF) and utility function of the DM over the values in X from Example 2 in the interval [4650; 5150]

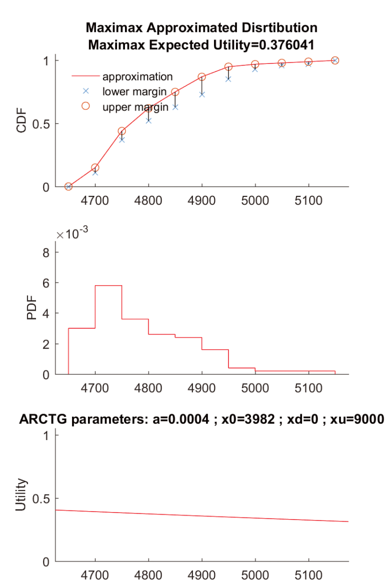

Figure 6. Graphics of $F_7^{p\neg W}(x)$, density (PDF) and utility function of the DM over the values in X from Example 2 in the interval [4650; 5150]

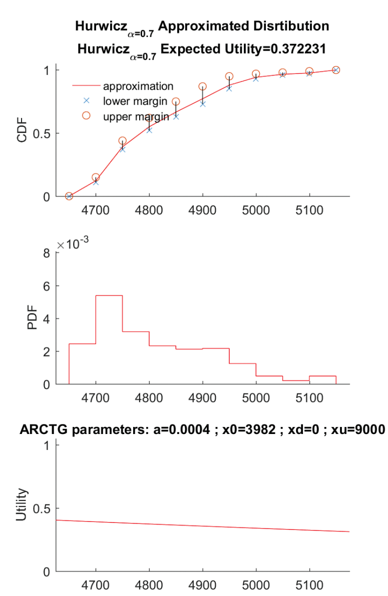

Figure 7. Graphics of $F_7^{pH_{0.7}}(x)$, density (PDF) and utility function of the DM over the values in X from Example 2 in the interval [4650; 5150]

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