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Orthogonal Approximation of Volterra Series and Wiener G-functionals Descriptions for Nonlinear Systems

K. Perev

Key Words: *Volterra series model; Wiener G-functionals; approximation of nonlinear systems; Legendre orthogonal polynomial series.*

Abstract. *This paper considers the problem of orthogonal series approximation of nonlinear systems described by Volterra series and G-functionals of Wiener. The Volterra series model is an input/output description of time-invariant nonlinear systems, where the Volterra kernels serve as generalizations for the linear system input response. Several model properties and conditions for the series convergence are presented. The Wiener G-functionals are orthogonal functions of time, where the input signal of the system is white gaussian noise. They describe nonhomogeneous input/output operators, i.e., operators where the change in the input signal level changes the level and the form of the output signal. The additional orthogonality introduced by Wiener, significantly simplifies their computation. The Volterra kernels and the Wiener G-functionals are approximated by orthogonal polynomials of Legendre. Legendre orthogonal polynomials are very effective when used for approximation of time functions on a finite interval of time. Formulas for computing the Fourier coefficients are developed. Several numerical examples for orthogonal series representation of order N = 1 for the nonlinear system with kernels up to third order are presented.*

1. Introduction

Nonlinear systems contain richer dynamics and more diverse properties than linear systems: existence of several equilibrium points, limit cycles, subharmonic oscillations, bifurcation points, chaotic behavior and others. The increased complexity of nonlinear behavior requires the development of specific nonlinear system methods and approaches, which deviate considerably from the corresponding linear system techniques. The main feature of the approaches for nonlinear system analysis is that there is no universal theory for all types of nonlinear problems. In order to explore different nonlinear phenomena, the researchers use different classes of nonlinear description models, which are related to different types of nonlinear system problems. One specific class that includes a very large number of physical systems is related to time invariant descriptions with finite memory. Such descriptions form the Wiener class of nonlinear system models and is implemented in terms of Volterra functional series. The Wiener theory for nonlinear systems is based on orthogonalization of a specific complete set of time invariant operators called Volterra operators [6].

Nonlinear system modeling with Volterra series was first proposed in [7] and was further extended in the works of Wiener [9, 10]. A rich bibliography sources for the application of the Volterra functional series model for analysis and design of nonlinear systems can be found in [6] and [5]. One of the main advantages of Volterra series models is the treatment of nonlinear systems in frequency domain. The so called generalized frequency response functions can be utilized for that purpose with the only restriction of using only finite number of Volterra terms by frequency truncation [4]. The order of the frequency domain Volterra series expansion refers to the order of the generalized frequency response function. Unfortunately, the derivation of high order Volterra series members is often computationally difficult task and requires extensive time and memory resources.

Volterra series models obey simple synthesis rules. The Wiener model for a stable causal nonlinear system can be derived by using linear system models by connecting them in terms of simple multiplicative structural relations. The Volterra series can be expanded as a polynomial functional series which resemble direct generalization of the linear convolution integral [2, 4]. Therefore, we can consider the Volterra series model as an input-output operator, where the kernels are multilinear generalizations of the impulse response function for linear systems [2]. The multilinear property of Volterra operators is this feature, which makes them particularly attractive for modeling nonlinear system behavior. In many nonlinear system applications, the usage of higher order Volterra series kernels is necessary in order to obtain a satisfactory truncation error, especially for severely nonlinear systems. Upper bounds for truncated Volterra series errors are provided in [2, 4], both in time and frequency domain, where conditions for series convergence are also discussed. In a recent publication, the application of Volterra series models for optimal model reduction of bilinear systems is discussed in [3]. The authors present a new framework for multipoint interpolation of the underlying Volterra series model and extend interpolation based conditions for optimal model reduction from the linear to the bilinear case. Model reduction for discrete-time bilinear systems is proposed in [8]. The presented method uses a frequency domain description, where the transfer function is expanded in terms of Laguerre functions basis. The Laguerre coefficients are computed recursively, thus avoiding cumbersome off-line computations.

This paper considers the problem of Volterra series model approximation. The Volterra series are presented in time domain by using high order Volterra kernel representations. In order to avoid certain convergence properties and measurement problems with respect to the Volterra kernels, the Wiener approach for using G-functionals is also presented. G-functionals of Wiener are orthogonal functions of time, when the input signal is a white gaussian noise. The Volterra as well as the Wiener kernels are approximated by using orthogonal polynomial series. The orthogonalization procedure is implemented by means of Legendre orthogonal series for difference with the widely used in practice Laguerre series representations. Legendre orthogonal series have the advantage of simple implementation, recursive procedure for the Fourier coefficients calculation, without using any weighting function under the integral, and finite definition interval, which is convenient for data points selection in practical implementations.

2. Preliminaries on the Volterra series description

The Volterra functional series is a basic model for nonlinear systems description. The Volterra model is an input/output model, where the nonlinear system is represented by the operator of Volterra as follows:

$$(1)\quad y(t) = H[u(t)] = \sum_{n=1}^{\infty} H_n\,[u(t)],$$

where $y(t)$ is the output signal, $u(t)$ is the input signal and $H_n(\cdot)$, $n = 1, 2, \cdots$ is the Volterra operator of order n. In the Volterra series model, H_n is an operator, which transforms the input signal into the output signal.

The relation (1) can be represented in time domain by the following expression:

$$(2)\quad y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n\,(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, d\tau_i,$$

where the functions $h_n(\tau_1, \cdots, \tau_n)$, $n = 1, 2, \cdots$ are called Volterra kernels. The Volterra kernels are a generalized description of the impulse response for linear systems. The Volterra functional series model gives the opportunity to use operator theory for exploring nonlinear systems. The usage of nonlinear operators leads to defining the main structural transformations by analogy to the linear case. Parallel connection between two nonlinear elements:

$$y(t) = H[u(t)] + G[u(t)],$$

multiplicative connection:

$$y(t) = H[u(t)]G[u(t)] \text{ and}$$

cascade connection:

$$y(t) = G\big[H[u(t)]\big] = (G * H)[u(t)],$$

where G and H are nonlinear operators [5].

The Volterra series model is a functional series model which is used for modeling nonlinear time-invariant systems with finite memory. The theory exploring such class of systems is called the Volterra-Wiener theory for nonlinear systems. Important properties of such class of systems are causality and stability [2, 6]. A system is causal, if for every input signal, the output system reaction in the current time moment does not depend on future values of the input signal.

We define the causality condition by the following expression:

$$(3)\quad P_T H_n P_T = P_T H_n,$$

where P_T is the truncation operator, whose action on arbitrary signal $x(t)$ is determined as follows:

$$(4)\quad P_T x(t) = x_T(t) = \begin{cases} x(t), & t \le T \\ 0, & t > T \end{cases}.$$

We can directly relate the causality property (3) to the physical realization of the system. Given system is physically realizable, if the past behavior uniquely determines the future behavior of the system. In terms of the Volterra kernels, the causality property of a given system is determined from the expression:

(5) $h_n(\tau_1, \tau_2, \cdots, \tau_n) = 0$ for all $\tau_i < 0$, $i = 1, 2, \cdots, n$.

Therefore, the n-th order Volterra kernel for a causal system will be different from zero only in the first quadrant of the n-th dimensional system space. Another important property of the explored nonlinear systems is the stability property. The Volterra series model is an input/output system model and therefore, we define naturally the stability property as the BIBO stability of the nonlinear system, which means that for a bounded input signal, the output signal is also bounded. The sufficient condition for nonlinear system stability is that, all Volterra kernels satisfy the integral relation:

(6) $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |h_n(\tau_1, \tau_2, \cdots, \tau_n)|\, d\tau_1 d\tau_2 \cdots d\tau_n < \infty,$
$n = 1, 2, \cdots.$

We can use the Volterra series model for frequency domain description of nonlinear systems. The nonlinear system transfer function is obtained from the Volterra kernels Laplace transform and the nonlinear system frequency response is computed from the Volterra kernels Fourier transform. The Laplace transform for the n-th order Volterra kernel is given by the following expression:

(7) $H_n(s_1, \cdots, s_n) = \int_0^{\infty} \cdots \int_0^{\infty} h_n(\tau_1, \cdots, \tau_n) \times$

$\times\, e^{-(s_1\tau_1 + \cdots + s_n\tau_n)} d\tau_1 \cdots d\tau_n.$

The n-th order frequency response is described as follows:

(8) $H_n(j\omega_1, \cdots, j\omega_n) = \int_0^{\infty} \cdots \int_0^{\infty} h_n(\tau_1, \cdots, \tau_n) \times$

$\times\, e^{-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)} d\tau_1 \cdots d\tau_n.$

We can compute the corresponding inverse Laplace and Fourier transforms by using the following expressions:

(9) $h_n(\tau_1, \cdots, \tau_n) = \frac{1}{(j2\pi)^n} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \cdots$

$\int_{\sigma_n - j\infty}^{\sigma_n + j\infty} H_n(s_1, \cdots, s_n) e^{(s_1\tau_1 + \cdots + s_n\tau_n)} ds_1 \cdots ds_n.$

(10) $h_n(\tau_1, \cdots, \tau_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots$

$\int_{-\infty}^{\infty} H_n(j\omega_1, \cdots, j\omega_n) e^{j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)} d\omega_1 \cdots d\omega_n.$

3. Legendre orthogonal polynomials

Orthogonal polynomials play an important role in approximation of continuous functions. Their application is based on the Weierstrass theorem, which claims that every continuous function defined on a bounded interval, can be approximated arbitrarily closely by a polynomial, whose order is determined by the required degree of accuracy. Therefore, every continuous function f from the Hilbert vector space $\mathcal{H}$ can be determined uniquely on the interval $[a, b]$ by the expression

$$f(t) = \sum_{n=0}^{\infty} d_n \psi_n(t),$$

where d_n, $n = 0, 1, \cdots$ are the Fourier coefficients of the function $f(t)$ with respect to the complete orthogonal system $\{\psi_n\}_{n=0}^{\infty}$.

We can normalize the orthogonal functions in order to fulfill the orthonormal condition:

(11) $\int_a^b \varphi_n(t)\varphi_m(t) dt = \begin{cases} 0 & for \quad m \neq n \\ 1 & for \quad m = n \end{cases}$,

where a and b can be arbitrary numbers or infinity. The Hilbert space under consideration is $L_2[a, b]$ and therefore, every approximated function $f \in L_2[a, b]$ satisfies the condition

$$\int_a^b f^2(t)\, dt < \infty.$$

We can approximate the function f in terms of orthogonal polynomial series as

$$f(t) \approx \sum_{n=0}^{N} d_n \psi_n(t),$$

where N is the order of polynomial approximation, and the Fourier coefficients d_n, $n = 0, 1, \cdots$ are calculated by the expression

$$d_n = \int_a^b f(t)\psi_n(t) dt.$$

The approximation error ε_N is determined from the expression [6]:

(12) $\varepsilon_N = \left[\int_a^b f^2(t) - \sum_{n=0}^{N} d_n^2\right]^{1/2}.$

One of the most frequently used orthogonal polynomial system for continuous functions approximation on the Hilbert space $L_2[a, b]$ is the system of Legendre orthogonal polynomials. The Legendre polynomials form a complete set of orthogonal polynomials in the Hilbert space $L_2[-1, 1]$ with constant weighting function $w(t) = 1$ on the interval $[-1, 1]$. The Legendre polynomials are obtained by applying the Gram-Schmidt orthogonalization procedure over the linearly independent set $\{1, t, t^2, \cdots\}$.

The Legendre polynomial of order n is represented by using the Rodrigues' formula as [1]:

(13) $P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n$, $n = 1, 2, \cdots.$

The Legendre polynomials satisfy also the recurrent relation:

(14) $P_{n+1}(t) = \frac{(2n+1)tP_n(t) - nP_{n-1}(t)}{n+1}$,

$P_0(t) = 1,$

$P_1(t) = t, \quad n = 1, 2, \cdots.$

The Legendre polynomials are transformed into Legendre functions in order to satisfy the orthonormal condition as follows:

$$\varphi_n = \sqrt{\frac{2n+1}{2}} P_n.$$

The first several Legendre polynomials on the interval $[-1, 1]$ are represented as follows:

$P_0(t) = 1,$

$P_1(t) = t,$

$P_2(t) = \frac{3}{2}\left(t^2 - \frac{1}{3}\right),$

$P_3(t) = \frac{5}{2}\left(t^3 - \frac{3}{5}t\right),$

$P_4(t) = \frac{35}{8}\left(t^4 - \frac{6}{7}t^2 + \frac{3}{35}\right),$

$P_5(t) = \frac{63}{8}\left(t^5 - \frac{10}{9}t^3 + \frac{5}{21}t\right).$

In many practical cases, the time interval for function approximation is different than $[-1, 1]$. In such cases, the Legendre functions are scaled and the variable of integration is changed. For example, the complete set of Legendre functions, defined on the interval $[0, T]$ are represented as follows:

(15) $\varphi_n(t) = \sqrt{\frac{2n+1}{2}} P_n(\tau), \ \tau = \frac{2}{T}t - 1, \ t \in [0, T].$

Then, every continuous function defined on the interval $[0, T]$ is approximated as:

(16) $f(t) \approx \sum_{n=0}^{N} c_n \sqrt{\frac{2n+1}{2}} P_n\left(\frac{2}{T}t - 1\right),$

where $c_n = \sqrt{\frac{2n+1}{2}} \int_0^T f(t) P_n\left(\frac{2}{T}t - 1\right) dt.$

4. Orthogonal approximation of nonlinear systems described by Volterra series

We consider nonlinear systems described in terms of Volterra series, whose kernels are approximated by orthogonal polynomial series. This result follows from the fact that, every functional, which describes the nonlinear system, can be represented by orthogonal functions series, defined on the interval under consideration. The so obtained series model is an analog of the Fourier series model, where the orthogonal system of functions can be obtained by using two different approaches. The first approach is to build the complete system of orthogonal functions by using the space of functions with vector argument. The second approach is to build the functional descriptions by using certain procedure of orthogonalization, and thus building the G-functionals of Wiener. We consider the first approach for orthogonalization, where the nonlinear system model is given by (1) in terms of the functionals (2). We assume that the system is stable and the first order kernel satisfies the condition

$$\int_0^\infty h_1^2(t) dt < \infty.$$

The Legendre series representation of this kernel on the interval $[0, T]$ can be obtained by the expression

$$h_1(t) = \sum_{n=0}^{\infty} c_n \varphi_n(t),$$

where the Fourier coefficients are determined from the integral

$$c_n = \int_0^T h_1(t) \varphi_n(t) dt.$$

The second order Volterra kernel $h_2(t_1, t_2)$ is a function of two variables. If the Volterra series convergence condition is satisfied

$$\int_0^\infty \int_0^\infty h_2^2(t_1, t_2) dt_1 dt_2 < \infty,$$

the Volterra kernel can be represented as

$$h_2(t_1, t_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c_{n_1 n_2} \varphi_{n_1}(t_1) \varphi_{n_2}(t_2),$$

where the Fouries coefficients are computed as follows:

$$c_{n_1 n_2} = \int_0^T \int_0^T h_2(t_1, t_2) \varphi_{n_1}(t_1) \varphi_{n_2}(t_2) dt_1 dt_2.$$

By analogy, the Volterra kernel of order p can be expanded in Legendre orthogonal series as:

(17) $h_p(t_1, \cdots, t_p)$

$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} c_{n_1 \cdots n_p} \varphi_{n_1}(t_1) \cdots \varphi_{n_p}(t_p).$$

with the corresponding Fourier coefficients computed as follows:

(18) $c_{n_1 \cdots n_p} =$

$$\int_0^T \cdots \int_0^T h_p(t_1 \cdots t_p) \varphi_{n_1}(t_1) \cdots \varphi_{n_p}(t_p) dt_1 \cdots dt_p.$$

Numerical example 1.

We consider the nonlinear time-invariant continuous-time system, described by the Volterra series model

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(t_1 \cdots t_n) \times$$

$$\times \prod_{i=1}^{n} u(t - t_i) dt_i,$$

where the Volterra kernels are given as $h_1(t_1) = e^{-t_1} 1(t_1)$, $h_2(t_1, t_2) = e^{-t_1} e^{-t_2} 1(t_1) 1(t_2)$ and so on.

The orthogonal expansion of the Volterra kernels in Legendre orthogonal series, where the order of approximation is $N = 1$, is obtained as follows:

$$h_1(t_1) = c_0 \varphi_0(t_1) + c_1 \varphi_1(t_2),$$

with the Fourier coefficients obtained by the expressions:

$$c_0 = \sqrt{\tfrac{1}{2}} \int_0^T e^{-t}\, dt,$$

$$c_1 = \sqrt{\frac{3}{2}} \int_0^T e^{-t} \left(\frac{2}{T} t - 1 \right) dt = \sqrt{\frac{3}{2}} \frac{2}{T} (1 - e^{-T} - T e^{-T}).$$

The second order Volterra kernel is approximated as follows:

$$h_2(t_1, t_2) = c_{00} \varphi_0(t_1) \varphi_0(t_2) + c_{01} \varphi_0(t_1) \varphi_1(t_2) + c_{10} \varphi_1(t_1) \varphi_0(t_2) + c_{11} \varphi_1(t_1) \varphi_1(t_2),$$

where the Fourier coefficients are computed as follows:

$$c_{00} = \sqrt{\tfrac{1}{2}} \sqrt{\tfrac{1}{2}} \int_0^T \int_0^T e^{-t_1} e^{-t_2} dt_1 dt_2 = \tfrac{1}{2} (1 - e^{-T})^2 ,$$

$$c_{01} = \sqrt{\tfrac{1}{2}} \sqrt{\tfrac{3}{2}} \int_0^T \int_0^T e^{-t_1}\, e^{-t_2} \left(\tfrac{2}{T} t_2 - 1 \right) dt_1 dt_2 = \tfrac{\sqrt{3}}{2} (1 - e^{-T}) \left((1 - e^{-T}) \left(\tfrac{2}{T} - 1 \right) - 2 e^{-T} \right),$$

$$c_{10} = \sqrt{\tfrac{1}{2}} \sqrt{\tfrac{3}{2}} \int_0^T \int_0^T e^{-t_1}\, e^{-t_2} \left(\tfrac{2}{T} t_1 - 1 \right) dt_1 dt_2 = \tfrac{\sqrt{3}}{2} (1 - e^{-T}) \left((1 - e^{-T}) \left(\tfrac{2}{T} - 1 \right) - 2 e^{-T} \right),$$

$$c_{11} = \sqrt{\frac{3}{2}} \sqrt{\frac{3}{2}} \int_0^T e^{-t_1} e^{-t_2} \left(\frac{2}{T} t_1 - 1 \right) \left(\frac{2}{T} t_2 - 1 \right) dt_1 dt_2 \;\; = \frac{3}{2} \left((1 - e^{-T}) \left(\frac{2}{T} - 1 \right) - 2 e^{-T} \right)^2 .$$

The orthogonal approximation of the third order Volterra kernel is obtained as follows:

$$\begin{aligned} h_3(t_1, t_2, t_3) = {} & c_{000} \varphi_0(t_1) \varphi_0(t_2) \varphi_0(t_3) + c_{001} \varphi_0(t_1) \varphi_0(t_2) \varphi_1(t_3) + c_{010} \varphi_0(t_1) \varphi_1(t_2) \varphi_0(t_3) \\ & + c_{011} \varphi_0(t_1) \varphi_1(t_2) \varphi_1(t_3) + c_{100} \varphi_1(t_1) \varphi_0(t_2) \varphi_0(t_3) + c_{101} \varphi_1(t_1) \varphi_0(t_2) \varphi_1(t_3) \\ & + c_{110} \varphi_1(t_1) \varphi_1(t_2) \varphi_0(t_3) + c_{111} \varphi_1(t_1) \varphi_1(t_2) \varphi_1(t_3) \end{aligned}$$

with the corresponding Fourier coefficients computed as follows:

$$c_{000} = \sqrt{\tfrac{1}{2}} \sqrt{\tfrac{1}{2}} \sqrt{\tfrac{1}{2}} \int_0^T \int_0^T \int_0^T e^{-t_1}\, e^{-t_2} e^{-t_3} dt_1 dt_2 dt_3 = \tfrac{1}{2\sqrt{2}} (1 - e^{-T})^3,$$

$$c_{001} = c_{010} = c_{100} = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} \sqrt{\frac{3}{2}} \int_0^T \int_0^T \int_0^T e^{-t_1} e^{-t_2} e^{-t_3} \left(\frac{2}{T} t_3 - 1 \right) dt_1 dt_2 dt_3 =$$

$$\tfrac{\sqrt{3}}{2\sqrt{2}} (1 - e^{-T})^2 \left[(1 - e^{-T}) \left(\tfrac{2}{T} - 1 \right) - 2 e^{-T} \right],$$

$$c_{011} = c_{101} = c_{110} = \sqrt{\frac{1}{2}} \sqrt{\frac{3}{2}} \sqrt{\frac{3}{2}} \int_0^T \int_0^T \int_0^T e^{-t_1} e^{-t_2} \left(\frac{2}{T} t_2 - 1 \right) e^{-t_3} \left(\frac{2}{T} t_3 - 1 \right) dt_1 dt_2 dt_3 =$$

$$\tfrac{3}{2\sqrt{2}} (1 - e^{-T}) \left[(1 - e^{-T}) \left(\tfrac{2}{T} - 1 \right) - 2 e^{-T} \right]^2,$$

$$c_{111} = \sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}\int_0^T\int_0^T\int_0^T e^{-t_1}\left(\frac{2}{T}t_1 - 1\right)e^{-t_2}\left(\frac{2}{T}t_2 - 1\right)e^{-t_3}\left(\frac{2}{T}t_3 - 1\right)dt_1dt_2dt_3 = \frac{3\sqrt{3}}{2\sqrt{2}}\left[\left(1 - e^{-T}\right)\left(\frac{2}{T} - 1\right) - 2e^{-T}\right]^3.$$

Analogically, we determine the orthogonal approximations with Legendre polynomial series the Volterra kernels of order four and higher.

5. Orthogonal approximation of nonlinear systems described by Wiener G-functionals

The main difficulties in describing nonlinear systems by Volterra series is the Volterra kernels measurement and the power series limited convergence properties. The measurement difficulty is related to the fact that measurement is possible only when each Volterra operator contribution can be separated from the total system response. However, there is no method for separating the Volterra operator contributions for infinite order systems. Another problem is concerned with the convergence properties of the Volterra series. It is well known that the Volterra series description is convergent only for a limited range of the system input amplitude. The convergence properties of the Volterra series model are similar to the convergence properties of Taylor series descriptions, since the Volterra series is Taylor series with memory. In order to overcome these difficulties, Wiener introduced a new type functionals, which are orthogonal for a certain type of input signals and are called G-functionals. G-functionals of Wiener are orthogonal functions of time, when the input signal is a white gaussian noise.

G-functionals of Wiener are nonhomogeneous, i.e., the input signal magnitude change leads not only to output signal magnitude change, but also the output signal form changes, leading to the following relation:

$$H_n[cx(t)] \neq c^n H_n[x(t)].$$

The Volterra operator of zero order is defined by the expression

$$H_0[x(t)] = h_0.$$

The nonhomogeneous Volterra functional of first order is defined as follows [6]:

$$(19)\quad g_1\left[h_1, h_{0(1)}; x(t)\right] = H_1[x(t)] + H_{0(1)}[x(t)].$$

The nonhomogeneous Volterra functional of second order is defined by the expression:

$$(20)\quad g_2\left[h_2, h_{1(2)}, h_{0(2)}; x(t)\right] = H_2[x(t)] + H_{1(2)}[x(t)] + H_{0(2)}[x(t)].$$

In the general case, the nonhomogeneous Volterra functional of arbitrary order is defined as follows:

$$(21)\quad g_p\left[h_p, h_{p-1(p)}, \cdots, h_{0(p)}; x(t)\right] = \sum_{n=0}^{p} H_{n(p)}\,[x(t)],$$

where $H_{n(p)}[x(t)]$ are the corresponding nonhomogeneous Volterra operators.

G-functionals of Wiener are a set of Volterra functionals

$$g_p\left[k_p, k_{p-1(p)}, \cdots, k_{0(p)}; x(t)\right],$$

for which the additional orthogonality conditions are satisfied

$$\varepsilon\left\{H_m[x(t)]g_n\left[k_n, k_{n-1,(n)}, \cdots, k_{0,(n)}; x(t)\right]\right\} = 0$$

for $m < n$,

where with $\varepsilon\{\cdot\}$ we denote the mean value of the expression in the middle parenthesis and $x(t)$ is a white Gaussian time function with autocorrelation $\phi_{xx}(t) = \alpha\delta(t)$.

For the G-functional of order zero, the following expression is valid:

$$G_0[k_0; x(t)] = k_0,$$

where k_0 is a constant, whose value depends on the input signal $x(t)$. The G-functional of first order is the corresponding nonhomogeneous Volterra operator of first order [6]:

$$(22)\quad g_1\left[k_1, k_{0(1)}; x(t)\right] = K_1[x(t)] + K_{0(1)}[x(t)],$$

where $K[x(t)]$ are Wiener operators with the additional orthogonality condition:

$$\varepsilon\left\{H_0[x(t)]g_1\left[k_1, k_{0(1)}; x(t)\right]\right\} = 0,$$

which is determined for a white gaussian noise as an input signal. This condition is determined from the relation:

$$(23)\quad \varepsilon\left\{H_0[x(t)]g_1\left[k_1, k_{0(1)}; x(t)\right]\right\} = h_0\int_{-\infty}^{\infty} k_1(\tau_1)\varepsilon\{x(t-\tau_1)\}\,d\tau_1 + h_0 k_{0(1)} = 0.$$

from which it follows that, $k_{0(1)} = 0$, since $\varepsilon\{x(t)\} = 0$. So, for the Wiener operator of first order we obtain the expression:

$$(24)\quad G_1[k; x(t)] = \int_{-\infty}^{\infty} k_1(\tau_1)\,x(t-\tau_1)d\tau_1.$$

The G-functional of second order is the corresponding nonhomogeneous Volterra operator of second order [6]:

$$(25)\quad g_2\left[k_2, k_{1(2)}, k_{0(2)}; x(t)\right] = K_2[x(t)] +$$

$$K_{1(2)}[x(t)] + K_{0(2)}[x(t)],$$

where the following additional conditions are satisfied:

$$\varepsilon\{H_0[x(t)]g_2[k_2, k_{1(2)}, k_{0(2)}; x(t)]\} = 0,$$
$$\varepsilon\{H_1[x(t)]g_2[k_2, k_{1(2)}, k_{0(2)}; x(t)]\} = 0.$$

It can be shown, that the first condition leads to condition for the Wiener kernel of zero order

$$k_{0(2)} = -\alpha \int_{-\infty}^{\infty} k_2(\tau, \tau) d\tau,$$

where α is the white noise intensity. From the second condition follows that $k_{1(2)}(\tau_1) = 0$. So, we obtain for the second order Wiener operator [6]:

$$(26)\quad G_2[k_2; x(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 - \alpha \int_{-\infty}^{\infty} k_2(\tau, \tau)\, d\tau.$$

Analogically, we obtain the Wiener operators of higher order. The output signal of the Wiener series model is obtained as follows:

$$y(t) = \sum_{n=0}^{\infty} G_n[k_n; x(t)],$$

where the nonzero kernels are as follows [6]: for the G_0 Wiener operator, it is k_0, for the G_1 Wiener operator it is k_1, for the G_2 they are $k_{0(2)}$ and k_2, for the G_3 they are $k_{1(3)}$ and k_3, for G_4 they are $k_{0(4)}$ and k_4, for the G_5 they are $k_{1(5)}$, $k_{3(5)}$ and k_5, etc., where the kernels are determined from the orthogonality conditions. It can be shown that, the derived Wiener kernels are obtained from the leading Wiener kernels as follows [6]:

$$k_{0(2)} = -\alpha \int_{-\infty}^{\infty} k_2(\tau, \tau) d\tau,$$

$$k_{1(3)}(\tau_1) = -3\alpha \int_{-\infty}^{\infty} k_3(\tau_1, \tau, \tau) d\tau,$$

$$k_{2(4)}(\tau_1, \tau_2) = -6\alpha \int_{-\infty}^{\infty} k_4(\tau_1, \tau_2, \tau, \tau) d\tau,\ k_{0(4)} = 3\alpha^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_4(\tau_1, \tau_1, \tau_2, \tau_2) d\tau_1\, d\tau_2,$$

$$k_{3(5)}(\tau_1, \tau_2, \tau_3) = -10\alpha \int_{-\infty}^{\infty} k_5(\tau_1, \tau_2, \tau_3, \tau, \tau) d\tau,$$

$$k_{1(5)} = 15\alpha^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_5(\tau_1, \tau_2, \tau_2, \tau_3, \tau_3) d\tau_2 d\tau_3, \text{ etc.,}$$

where α is the white noise intensity. The relation between Wiener and Volterra kernels are given as follows [6]:

$$h_0 = k_0 + k_{0(2)} + k_{0(4)},\ h_1 = k_1 + k_{1(3)} + k_{1(5)},$$

$$h_2 = k_2 + k_{2(4)},\ h_3 = k_3 + k_{3(5)},\ h_4 = k_4,\ h_5 = k_5, \text{ etc.}$$

It is clear that, all Wiener kernels with indices in brackets (derived Wiener kernels) can be obtained from kernels with indices without brackets (leading Wiener kernels). Therefore, it is necessary to obtain orthogonal series development only for Wiener kernels with indices without brackets. For obtaining the Legendre orthogonal series representation for Wiener operators, we can use the Legendre orthogonal series representations for Volterra operators. In this sense, $k_3 = h_3 - k_{3(5)}$ and since $k_5 = h_5$, we obtain:

$$k_3(\tau_1, \tau_2, \tau_3) = h_3(\tau_1, \tau_2, \tau_3) + 10\alpha \int_{-\infty}^{\infty} h_5(\tau_1, \tau_2, \tau_3, \tau, \tau) d\tau.$$

We assume that the system is causal and stable. With respect to the Wiener kernels, these conditions are determined as:

$$k_n(\tau_1, \cdots, \tau_n) = 0,$$

for every $\tau_i < 0$, $i = 0, 1, 2, \cdots$ (the causality condition) and also

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |k_n(\tau_1, \cdots, \tau_n)| d\tau_1 \cdots d\tau_n < \infty,$$

for $n = 0, 1, 2, \cdots$ (the stability condition).

Then, every Wiener operator can be represented by Legendre orthogonal polynomial series on the interval $[0, T]$ as follows [6]:

$$(27)\quad k_p(\tau_1, \cdots, \tau_p) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} c_{n_1 \cdots n_p} \varphi_{n_1}(\tau_1) \cdots \varphi_{n_p}(\tau_p),$$

where the Fourier coefficients are computed as follows:

$$(28)\quad c_{n_1 \cdots n_p} = \int_0^T \ldots \int_0^T k_p(\tau_1, \cdots, \tau_p) \varphi_{n_1}(\tau_1) \cdots \varphi_{n_p}(\tau_p) d\tau_1 \cdots d\tau_p.$$

The Wiener operator are determined as follows [6]:

$$(29)\quad G_p[k_p; x(t)] = G_p\left[\sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} c_{n_1 \cdots n_p} \varphi_{n_1}(\tau_1) \cdots \varphi_{n_p}(\tau_p); x(t)\right].$$

where due to the linearity property of the parallel connection, we can write:

$$(30)\quad G_p[k_p; x(t)] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} c_{n_1 \cdots n_p} G_p\left[\varphi_{n_1}(\tau_1) \cdots \varphi_{n_p}(\tau_p); x(t)\right].$$

In order to determine the zero order kernel, we can obtain:

$k_0 = \bar{y}(t)$

or this is the mean value of the output signal, where the input signal is white Gaussian noise with intensity α. The Wiener kernel of first order can be obtained as

$k_1(t) = \sum_{n=0}^{\infty} c_n \, \varphi_n(t),$

where $c_n = \int_0^T k_1(t)\varphi_n(t)dt$.

For the second order Wiener kernel, we obtain

$k_2(\tau_1, \tau_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} c_{n_1 n_2} \, \varphi_{n_1}(\tau_1)\varphi_{n_2}(\tau_2),$

for the Fourier coefficient of second order, we obtain:

$c_{n_1 n_2} = \int_0^T \int_0^T k_2(\tau_1, \tau_2) \, \varphi_{n_1}(\tau_1)\varphi_{n_2}(\tau_2) d\tau_1 d\tau_2,$

etc.

Numerical example 2.

We consider the nonlinear time-invariant continuous-time system, described by the Wiener G-functionals

$y(t) = \sum_{n=0}^{\infty} G_n[k_n; x(t)].$

The Wiener kernels are given as follows:

$k_1(\tau_1) = e^{-\tau_1} 1(\tau_1),$

$k_2(\tau_1, \tau_2) = e^{-\tau_1} e^{-2\tau_2} 1(\tau_1) 1(\tau_2),$

$k_3(\tau_1, \tau_2, \tau_3) = e^{-\tau_1} e^{-2\tau_2} e^{-3\tau_3} 1(\tau_1) 1(\tau_2) 1(\tau_3),$

etc.

We consider the Legendre orthogonal series representation of Wiener kernels, where the approximation order is $N = 1$.

$k_p(\tau_1, \cdots, \tau_p) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_p=0}^{\infty} c_{n_1 \cdots n_p} \varphi_{n_1}(\tau_1) \cdots \varphi_{n_p}(\tau_p),$

For example, the first order Wiener kernel in Legendre orthogonal polynomials series is determined as follows:

$k_1(\tau_1) = c_0 \varphi_0(\tau_1) + c_1 \varphi_1(\tau_1),$

where the Fourier coefficients are determined as follows:

$c_0 = \sqrt{\frac{1}{2}} \int_0^T e^{-\tau} \, d\tau = \sqrt{\frac{1}{2}} (1 - e^{-T}),$

$c_1 = \sqrt{\frac{3}{2}} \int_0^T e^{-\tau} \left(\frac{2}{T}\tau - 1\right) d\tau = \sqrt{\frac{3}{2}} \frac{2}{T} (1 - e^{-T} - T e^{-T}).$

The second order Wiener kernels developed in Legendre orthogonal polynomials series are obtained as follows:

$$k_2(\tau_1, \tau_2) = c_{00}\varphi_0(\tau_1)\varphi_0(\tau_2) + c_{01}\varphi_0(\tau_1)\varphi_1(\tau_2) + c_{10}\varphi_1(\tau_1)\varphi_0(\tau_2) + c_{11}\varphi_1(\tau_1)\varphi_1(\tau_2).$$

The corresponding Fourier coefficients are computed as follows:

$$c_{00} = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} \int_0^T \int_0^T e^{-\tau_1} e^{-2\tau_2} \, d\tau_1 d\tau_2 = \frac{1}{4} (1 - e^{-T})(1 - e^{-2T}),$$

$$c_{01} = \frac{\sqrt{3}}{2} \int_0^T \int_0^T e^{-\tau_1} e^{-2\tau_2} \left(\frac{2}{T}\tau_2 - 1\right) d\tau_1 d\tau_2 = \frac{\sqrt{3}}{4T} (1 - e^{-T})(1 - T - (1 + T)e^{-2T}),$$

$$c_{10} = \frac{\sqrt{3}}{2} \int_0^T \int_0^T e^{-\tau_1} \left(\frac{2}{T}\tau_1 - 1\right) e^{-2\tau_2} d\tau_1 d\tau_2 = \frac{\sqrt{3}}{4} (1 - e^{-2T})(2 - T - (2 + T)e^{-T}),$$

$$c_{11} = \frac{3}{2} \int_0^T \int_0^T e^{-\tau_1} \left(\frac{2}{T}\tau_1 - 1\right) e^{-2\tau_2} \left(\frac{2}{T}\tau_2 - 1\right) d\tau_1 d\tau_2 = \frac{3}{4T^2} (2 - T - (2 + T)e^{-T})(1 - T - (1 + T)e^{-T}).$$

The third order Wiener kernels developed in Legendre orthogonal polynomials series are obtained as follows:

$$\begin{aligned} k_3(t_1, t_2, t_3) = {} & c_{000}\varphi_0(t_1)\varphi_0(t_2)\varphi_0(t_3) + c_{001}\varphi_0(t_1)\varphi_0(t_2)\varphi_1(t_3) + c_{010}\varphi_0(t_1)\varphi_1(t_2)\varphi_0(t_3) \\ & + c_{011}\varphi_0(t_1)\varphi_1(t_2)\varphi_1(t_3) + c_{100}\varphi_1(t_1)\varphi_0(t_2)\varphi_0(t_3) + c_{101}\varphi_1(t_1)\varphi_0(t_2)\varphi_1(t_3) \\ & + c_{110}\varphi_1(t_1)\varphi_1(t_2)\varphi_0(t_3) + c_{111}\varphi_1(t_1)\varphi_1(t_2)\varphi_1(t_3) \end{aligned}$$

The corresponding Fourier coefficients are computed as follows:

$$c_{000} = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} \int_0^T \int_0^T \int_0^T e^{-\tau_1} \, e^{-2\tau_2} e^{-3\tau_3} d\tau_1 d\tau_2 d\tau_3 = \frac{1}{12\sqrt{2}} (1 - e^{-T})(1 - e^{-2T})(1 - e^{-3T}),$$

$$c_{001}=\frac{\sqrt{3}}{2\sqrt{2}}\int_0^T\int_0^T\int_0^T e^{-\tau_1}e^{-2\tau_2}\,e^{-3\tau_3}\left(\frac{2}{T}\tau_3-1\right)d\tau_1 d\tau_2 d\tau_3=\frac{\sqrt{3}}{4\sqrt{2}}(1-e^{-T})(1-e^{-2T})\left[\frac{2}{9T}(1-e^{-3T})-\frac{2}{3}e^{-3T}\right],$$

$$c_{010}=\frac{\sqrt{3}}{2\sqrt{2}}\int_0^T\int_0^T\int_0^T e^{-\tau_1}\,e^{-2\tau_2}\left(\frac{2}{T}\tau_2-1\right)e^{-3\tau_3}d\tau_1 d\tau_2 d\tau_3=\frac{1}{2\sqrt{6}}(1-e^{-T})(1-e^{-2T})\left[\frac{1-T}{2T}(1-e^{-2T})-e^{-2T}\right],$$

$$c_{011}=\frac{3}{2\sqrt{2}}\int_0^T\int_0^T\int_0^T e^{-\tau_1}e^{-2\tau_2}\left(\frac{2}{T}\tau_2-1\right)e^{-3\tau_3}\left(\frac{2}{T}\tau_3-1\right)\,d\tau_1 d\tau_2 d\tau_3=\frac{3}{2\sqrt{2}}(1-e^{-T})\times$$

$$\times\left[\frac{1-T}{2T}(1-e^{-2T})-e^{-2T}\right]\left[\frac{2-3T}{9T}(1-e^{-3T})-\frac{2}{3}e^{-3T}\right],$$

$$c_{100}=\frac{\sqrt{3}}{2\sqrt{2}}\int_0^T\int_0^T\int_0^T e^{-\tau_1}\left(\frac{2}{T}\tau_1-1\right)e^{-2\tau_2}e^{-3\tau_3}d\tau_1 d\tau_2 d\tau_3=\frac{1}{4\sqrt{6}}(1-e^{-2T})(1-e^{-3T})\left[\frac{2-T}{T}(1-e^{-T})-2e^{-T}\right],$$

$$c_{101}=\frac{3}{2\sqrt{2}}\int_0^T\int_0^T\int_0^T e^{-\tau_1}\left(\frac{2}{T}\tau_1-1\right)e^{-2\tau_2}e^{-3\tau_3}\left(\frac{2}{T}\tau_3-1\right)d\tau_1 d\tau_2 d\tau_3=\frac{3}{4\sqrt{2}}(1-e^{-T})\times$$

$$\times\left[\frac{2-T}{T}(1-e^{-T})-2e^{-T}\right]\left[\frac{2-3T}{9T}(1-e^{-3T})-\frac{2}{3}e^{-3T}\right],$$

$$c_{110}=\frac{3}{2\sqrt{2}}\int_0^T\int_0^T\int_0^T e^{-\tau_1}\left(\frac{2}{T}\tau_1-1\right)e^{-2\tau_2}\left(\frac{2}{T}\tau_2-1\right)e^{-3\tau_3}\;d\tau_1 d\tau_2 d\tau_3=\frac{1}{2\sqrt{2}}(1-e^{-3T})\times$$

$$\times\left[\frac{2-T}{T}(1-e^{-T})-2e^{-T}\right]\left[\frac{1-T}{2T}(1-e^{-2T})-e^{-2T}\right],$$

$$c_{111}=\frac{3\sqrt{3}}{2\sqrt{2}}\int_0^T\int_0^T\int_0^T e^{-\tau_1}\left(\frac{2}{T}\tau_1-1\right)e^{-2\tau_2}\left(\frac{2}{T}\tau_2-1\right)e^{-3\tau_3}\left(\frac{2}{T}\tau_3-1\right)d\tau_1 d\tau_2 d\tau_3=$$

$$\frac{3\sqrt{3}}{2\sqrt{2}}\left[\frac{2-T}{T}(1-e^{-T})-2e^{-T}\right]\left[\frac{1-T}{2T}(1-e^{-2T})-e^{-2T}\right]\left[\frac{2-3T}{9T}(1-e^{-3T})-\frac{2}{3}e^{-3T}\right].$$

6. Conclusion

The paper considers the problem of orthogonal approximation of the Volterra and Wiener kernels for nonlinear system descriptions. The Volterra series model for describing nonlinear systems is presented and its convergence properties are discussed. The paper considers also the Wiener G-functionals representation for nonlinear system modeling. The orthogonal approximation of the presented models is developed in time domain by using the Legendre orthogonal polynomials. The basic advantages of using the Legendre series model are its computational efficiency in terms of simple recurrence relations and the possibility for approximation continuous functions on a finite interval of time. Formulas for computing the Fourier coefficients in the orthogonal series representations are developed. Numerical examples for the orthogonal series model of order $N=1$ for the nonlinear system with kernels up to third order are presented.

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__Kamen Perev__ received the Engineering degree in Automatics from VMEI – Sofia in 1985. He received the M.S. degree in Control systems and digital signal processing and the Ph.D. degree in Control systems, both from Northeastern University, Boston, MA, USA in 1991 and 1993, respectively. He became an Assistant Professor at the Technical University of Sofia in 1997. In 2006, he was promoted to Associate Professor at the same university. Associate Professor Perev defended dissertation entitled "Generalized method for model reduction of high order plants" for the Sc.D. degree in 2019. His area of scientific interests includes linear and nonlinear control systems, robust control theory, model order reduction and system approximation. He is a member of the Bulgarian Union of Automatics and Informatics (SAI) and the Technical Committee 2.2 on Linear Control Systems of IFAC.

Contacts:
Systems and Control Department
Technical University of Sofia
8 "Kl. Ohridski" Blvd.
1756 Sofia, Bulgaria
e-mail: kperev@tu-sofia.bg