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Investigation of Heuristic Algorithms for One Dimensional Optimization with Accelerated Convergence

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Key Words: *One-dimensional optimization; accelerated convergence; dichotomy; double dichotomy; algorithm efficiency; uncertainty interval.*

Abstract. *A new heuristic optimization algorithm with accelerated convergence is proposed for search a maximum of one-dimensional (single variable), unimodal objective functions. The algorithm is a combination of the dichotomy method, the Kiefer–Johnson method, and the fourth grade functional series. A comparative analysis has been made with other known methods and its effectiveness and accelerated convergence have been demonstrated for cases where the uncertainty interval in the search is very large. The efficiency of the algorithm compared to other known algorithms is based on the number of the objective function evaluation to find the optimum with different accuracy requirements for localization the maximum (or minimum) of the function. The method and proposed algorithm is suitable for parameters estimation in mathematical models.*

Introduction

The task of one-dimensional optimization for unimodal objective functions is considered to be solved but still there are practical problems that need to increase the speed of convergence of methods to find the optimal solution with certain accuracy. Such problems are encountered in the task of parameters estimation in complex mathematical models of technological processes and systems for which the boundaries of the region of search are very large or not given at all. Often, it is necessary to specify the unknown parameters in multidimensional systems to be hierarchically defined in a certain sequence of the parameters priorities in the model, thus requiring the multidimensional optimization task to be decomposed into a sequence of one-dimensional tasks. In many of the algorithms for multidimensional optimization, the search of maximum (or minimum) of the objective function is in a gradient, random or other rational direction of multidimensional space. This also requires choosing an optimal one-dimensional strategy in a multidimensional parametric space.

Solving such optimization tasks can be accomplished by creating algorithms to find the optimum sequence of reducing the uncertainty interval when searching for the optimal solutions with unimodal objective functions

$$Q(x) \rightarrow \max_{x}, X \in [A, B].$$

To solve the problem of one-dimensional optimization, the methods of scanning, dichotomous, golden section search, the Kiefer – Johnson method using the Fibonacci numbers, interpolation methods of Davidon, Fletcher, Powell and other methods are applied. [2,3,4,5]. In order to increase the speed of convergence, the research effort in recent years is directed to optimal combinations of different optimization methods using heuristic rules. A method for one-dimensional optimization, which combines the positive features of the dichotomous search followed by the Kiefer–Johnson method using the Fibonacci numbers, is proposed in [1,2]. The algorithm excels in convergence speed all one-dimensional methods created so far for unimodal objective (functions, with a given precision of the desired optimal solution.

In this article a new combined algorithm applying three sequences of functional series of numbers is proposed which significantly accelerate the convergence in search for extremum.

Strategy of the Combined Series

The strategy of the combined method is based on the good convergence of algorithms studied [1,2] and offering a new combined functional series of numbers (order) and heuristic rules. The basic heuristic rules that can be formulated in a one-dimensional search are the following:

(1) An effective optimization strategy is the one, which achieves the maximum reduction in the uncertainty interval regarding to one evaluation of the objective function in the optimization algorithm.

(2) If there are two calculations of the unimodal function Q(x) at two points of the control variable x, $Q(X_1)$ and $Q(X_2)$, the rejected area is determined by the worse result obtained at X_1 or X_2. If $Q(X_1) = Q(X_2)$, the uncertainty range remains between X_1 and X_2, $X \in (X_1, X_2)$.

(3) At every shortening of the allowable interval that contains the optimal value of the objective function, the convergence rate of the algorithm for an unimodal function depends on the heuristic constant L_{Δ} for a multiplicity of the decrease or increase of the search step parameter by the control parameter *x*.

(4) A fast converging algorithm can be expected when combining the strengths of several search algorithms [3].

In the present article an algorithm with accelerated convergence is proposed, which combines the positive qualities of the following methods: Kiefer-Johnson method, using the Fibonacci numbers, second-order rank functional series ("dichotomy") and one new fourth-order rank functional series, called ”double dichotomy”, not used till now in one-dimensional optimization strategies.

The combined algorithm starts with a "double dichotomy" (shortly referred as "4"), which quickly reject a large part of the uncertainty interval, switching to "dichotomy" (labeled with "2") and finally ending with Fibonacci numbers (labeled with "F"). Comparison with other one-dimensional search algorithms is done using the number of calculations S of the objective function Q(X) for a given set of different accuracy of localization of the optimal solution X_{max}, $Q_{max} = Q(X_{max})$.

The algorithm of the method is as follows:

1. Setting the objective function Q(X), which we assume will be maximized.

2. Setting the initial (A) and the final (B) limits of the uncertainty interval of the control parameter $X \in [A,B]$ and the required absolute accuracy Δ_{min} of the control parameter X to locate the maximum of Q(X).

3. Set S = 0 for the identifier for number of calculations S of the objective function Q(X). Set K = 1 for the identifier for the sequence of depletion of the numbers from the combined series Rn.

4. Calculate the accessory number M

$M=(B-A)/ \Delta_{min}$.

5. A combined series that includes the first 6 numbers of the Fibonacci numbers and the second and fourth orders functional series (F–2–4) is created with the following algorithm:

5.1. $R_1 = R_2 = 1$;

5.2. $R_i = R_{i-2} + R_{i-1}$, for $2 \le i \le 6$, in satisfying the condition: $R_i < M$;

5.3. $R_i = 2.R_{i-1}$, for $7 \le i \le R_{sw}$, in satisfying the condition: $R_i < M$;

5.4. $R_i = 4.R_{i-1}$, for $i > R_{sw}$, until is finding the number of combined series, which correspond to condition $R_n \ge M$.

If the number of combined series, which correspond to condition $R_n \ge M$ is found for condition $2 \le i \le 6$, K is set to 2.

To find the optimal number for switching the algorithm, the following values of R_{sw} = 7, 8, 9 and 10 are investigated in the present article.

6. Calculate the actual accuracy for extremum localization

$\Delta_m=(B-A)/R_n$.

7. Calculate the objective function at the beginning of the interval (X = A) and assume to Q_{max}:Q_{max} = Q (A).S = S + 1. The value of A assumes to X_{max}: X_{max} = A, K = 1.

8. Make a step with the number of the combined series in the direction of x increasing and calculate the value of X_1:$X_1 = X_{max} + \Delta_m$.R[N-K] .

9. Calculate the function $Q(X_1)$ at point X_1:$Q_1 = Q(X_1)$. S = S + 1.

10. If the step in (8) is successful ($Q_1 > Q_{max}$), the value obtained for Q_1 assigns to Q_{max} and the value of X_1 assigns to X_{max}. Make a new step in the same direction (straight direction) with the same number of the combined order. The algorithm continues from point (9).

11. If the value of X_1 coincides with the upper limit of the interval of control parameter (X_1 = B), the value of the objective function calculated in this point compares with the resulting maximum value (Q_{max}):

11.1. If the step is successful ($Q_1 > Q_{max}$), make the next step from the upper limit (B) of the interval of changing of the control parameter X in the opposite direction using the next number of the combined series R[i], i = N, N-1, N-2, …, 3, 2, 1: $X_1 = B-\Delta_m$.R[i]

11.1.1. Calculate the value of the objective function at this point $Q_1 = Q(X_1)$, S=S+1. Q_1 is compared with the current maximum value obtained (Q_{max});

11.1.2. If the step is successful ($Q_1 > Q_{max}$), make a new step in the same direction (i.e., in the opposite direction) with the same number of the combined series R[i] used in 11.1. The algorithm proceeds from point (11.1.1);

11.1.3. If the step is unsuccessful ($Q_1 \le Q_{max}$), the algorithm proceeds from point (8).

11.2. If the step is unsuccessful ($Q_1 \le Q_{max}$), the algorithm continues from point (13).

12. If the first step is unsuccessful, make a new step from A in the same direction with the next number of the combined series R[i], i = N, N-1, N-2,…,3, 2, 1: $X_1 = X_{max} - \Delta_m$.R[N-K] and the algorithm continues from point (9).

13. If the step at point (8) is unsuccessful, make a step in the opposite direction with the next number of the combined series R[i], i = N, N-1, N-2,…,3, 2, 1: $X_1 = X_{max} - \Delta_m$.R[i]

14. If $X_1 \le A$, make a new step from X_1 by the same equation p. (13), in the same direction with the next number from combined series R[i], i = N, N-1, N-2, …3, 2, 1: $X_1 = X_{max} - \Delta_m$.R[N-K]. The algorithm continues from point (15).

15. Calculate the function $Q(X_1)$ at point X_1: $Q_1 = Q(X_1)$. S = S + 1.

16. If the step in point (13) is successful ($Q_1 > Q_{max}$), the value, obtained for Q_1 assigns to Q_{max} and the value of X_1 assigns to X_{max}. Make a new step in the same direction with the same number of the combined order. The algorithm continues from point (14).

17. If the step in point (13) is unsuccessful ($Q_1 \le Q_{max}$), the algorithm proceeds from point (8).

18. The algorithm terminates when all the numbers of the generated combined series are depleted.

Formation of Combined Series

The formation of the combined series is shown in *table 1*. Switching from the “Fibonacci” numbers to the order "2" takes place under number 6. Part of the possible switches from series "2" to series "4", during combined series creation, are given in *table 1* with thick arrows.

One of the possible shifts from series "4" to series "2" and to “Fibonacci”, using the combined series for optimization, is given for illustration with dotted arrows in *table 1*.

Table 1. Formation of combined series

No	“Fibonacci”	“2” Ni	“4” Ni
1	**1**	1	1
2	**1**	2	4
3	**2**	4	**16**
4	**3**	**8**	**64**
5	**5**	**16**	**256**
6	**8**	32	**1024**
7	13	**64**	**4096**
8	21	128	**16384**
9	34	**256**	**65536**
10	55	512	262144
11	89	**1024**	1048576
12	144	2048	4194304
13	233	**4096**	16777216
14	377	8192	67108864
15	610	**16384**	268435456
16	987	32768	1073741824
17	1597	**65536**	4294967296

Investigation of the Strategy of the Combined Series

In the present study for creation and using of the combined series are studied a combined series of numbers of Fibonacci and fourth degree functional series and also algorithm with only fourth degree functional series.

The following indications are adopted in the tables and figures presented below, which illustrates the comparative analysis:

Δ_{min} – absolute precision set to locate extremum on the control parameter x ϵ [A,B] in the uncertainty range [A, B];

Δ_m – actual absolute precision set to locate extremum on the control parameter X with studied algorithm;

lg(Δ_m/(B-A)) – logarithm of actual relative accuracy;

S – number of the objective function evaluation for the optimal solution finding with accuracy Δ_m.

The following notations are accepted for the 10 studied algorithms:

D “Dichotomy method”;

Z „Golden section” method;

F Method of Kiefer – Johnson using Fibonacci numbers (KJF);

2-F Combined method of “dichotomy” and “KJF” with switching number 8 from “dichotomy” to “KJF” with following series: 1, 1, 2, 3, 5, <u>8</u>, 16, 32, 64, 128, 256, 512, 1024, ...;

4-2-F-(1) Combined method of “double dichotomy”, “dichotomy” and “KJF” with switching numbers 128 and 8 with following series: 1, 1, 2, 3, 5, <u>8</u>, 16, 32, 64, <u>128</u>, 512, 2048, 8192,…;

4-2-F-(2) Combined method of “double dichotomy”, “dichotomy” and “KJF” with switching numbers 64 and 8: 1, 1, 2, 3, 5, <u>8</u>, 16, 32, <u>64</u>, 256, 1024, 4096, 16384, …;

4-2-F-(3) Combined method of “double dichotomy”, “dichotomy” and “KJF” with switching numbers 32 and 8: 1, 1, 2, 3, 5, <u>8</u>, 16, <u>32</u>, 128, 512, 2048, 8192,...;

4-2-F-(4) Combined method of “double dichotomy”, “dichotomy” and “KJF” with switching numbers 16 and 8: 1, 1, 2, 3, 5, <u>8</u>, <u>16</u>, 64, 256, 1024, 4096, 16384, …;

4-F Combined method of “double dichotomy” and “KJF” with switching number 8 with following series: 1, 1, 2, 3, 5, <u>8</u>, 32, 128, 512, 2048, 8192,…;

4 A method of “double dichotomy” with following series: 1, 4, 16, 64, 256, 1024, 4096, 16384,…

To investigate the effectiveness of all algorithms, an illustrative, unimodal objective function is used Q(x)= $-3x^2+21{,}6x+1$, for which the maximum is known $Q_{max}(X^*)$ = 39.88 at X_{max} = 3.6.

The convergence to the optimal solution is studied for all algorithms, with the desired absolute precision Δ_{min} from 1.10^{-1} to 1.10^{-9} of the interval [B-A] for localization of maximum of Q(X), by counting the number of calculations S of the objective function Q(X) for each accuracy in the uncertainty intervals and each algorithm being studied.

Three uncertainty intervals have been studied [A, B]:

(1) x ϵ [0,20] (basic interval);

(2) x ϵ [0,2000] (increased 100 times basic interval);

(3) x ϵ [0,200 000] (increased 10000 times basic interval).

The obtained results for combined series 4-2-F-(4) as an example for the interval x ϵ [0,20] are shown in *table 2*.

Table 2. Results for combined series for “double dichotomy”, “dichotomy” and “F” (4-2-F-(4), x ϵ [0,20]

Δ min	Δ m	Δm /(B – A)	S	Qm	Xm
1.10^{-1}	0.078125000000	0.003906250000000	**11**	39.8798828125	3.5937500000
1.10^{-2}	0.004882812500	0.000244140625000	**15**	39.8799943924	3.5986328125
1.10^{-3}	0.000305175781	0.000015258789062	**17**	39.8799999356	3.5998535156
1.10^{-4}	0.000076293945	0.000003814697266	**23**	39.8799999999	3.6000061035
1.10^{-5}	0.000004768372	0.000000238418579	**27**	39.8800000000	3.6000013351
1.10^{-6}	0.000000298023	0.000000014901161	**29**	39.8800000000	3.6000001431
1.10^{-7}	0.000000074506	0.000000003725290	**35**	39.8800000000	3.5999999940
1.10^{-8}	0.000000004657	0.000000000232831	**39**	39.8800000000	3.5999999987
1.10^{-9}	0.000000000291	0.000000000014552	**41**	39.8800000000	3.5999999999

For the comparative analysis of the algorithms the graphical representation of the dependence S = *f*(precision) is used. The precision increasing is expressed by the logic sequence (1, 2, 3,..., 9) which corresponds to the absolute value of the logarithm of the given (or achieved) precision $|lg(\Delta_{min})|$.

The average number of calculations of the objective function ($S_{av,}$ without rounding) with a given precision Δ to locate the maximum are given in *table 3*. The average number of objective function calculations (S_d) at achieved precision lg(Δm) are given in *table 4*. The average number of calculations of the objective function (S_{A-B}) with relative accuracy achieved lg(Δm/(B-A)) is given in *table 5*.

Table 3. Average number of calculations of objective function (S_{av}) with a given accuracy Δm for maximum location

Interval	**Sav for an algorithm**									
[A – B]	D	Z	F	2-F	4-2-F (1)	4-2-F (2)	4-2-F (3)	4-2-F (4)	4-F	4
[0 - 20]	43.5	31.7	30.3	25.3	27.2	27.0	26.3	26.3	25.0	24.4
[0 - 2000]	53.0	41.2	40.9	32.0	33.8	33.7	33.0	33.4	31.3	32.7
[0 - 200000]	64.3	50.8	50.7	38.7	34.6	34.8	34.1	33.7	32.6	33.3

Table 4. Average number of calculations of objective function (S_d) at achieved precision lg(Δm) for maximum localization with different algorithms and under different intervals of searching

Interval	**Sd for an algorithm**									
[A - B]	D	Z	F	2-F	4-2-F (1)	4-2-F (2)	4-2-F (3)	4-2-F (4)	4-F	4
[0 - 20]	48.2	35.4	33.4	27.6	28.8	28.8	28.2	28.2	27.0	25.8
[0 - 2000]	61.8	47.4	43.8	34.0	36.0	35.4	35.0	34.8	33.4	31.6
[0 - 200000]	70.0	50.7	49.8	38.3	34.3	33.8	33.3	32.3	31.3	30.0

Table 5. Average number of calculations of the objective function (S_{A-B}) with relative accuracy achieved lg(Δm/(B-A)) for maximum localization with different algorithms and at different intervals of searching

Interval	**SA-B for an algorithm**									
[A - B]	D	Z	F	2-F	4-2-F (1)	4-2-F (2)	4-2-F (3)	4-2-F (4)	4-F	4
[0 - 20]	41.0	30.0	28.2	23.6	25.2	24.2	24.4	23.6	23.4	23.1
[0 - 2000]	53.4	39.2	37.8	30.6	30.8	30.8	30.4	30.4	29.0	28.0
[0 - 200000]	74.3	53.3	46.5	39.8	35.8	35.8	35.7	33.8	32.7	31.5

The graphical interpretation of the results from *table 3*, *table 4* and *table 5* are given in *figure 1*, *figure 2* and *figure 3*.

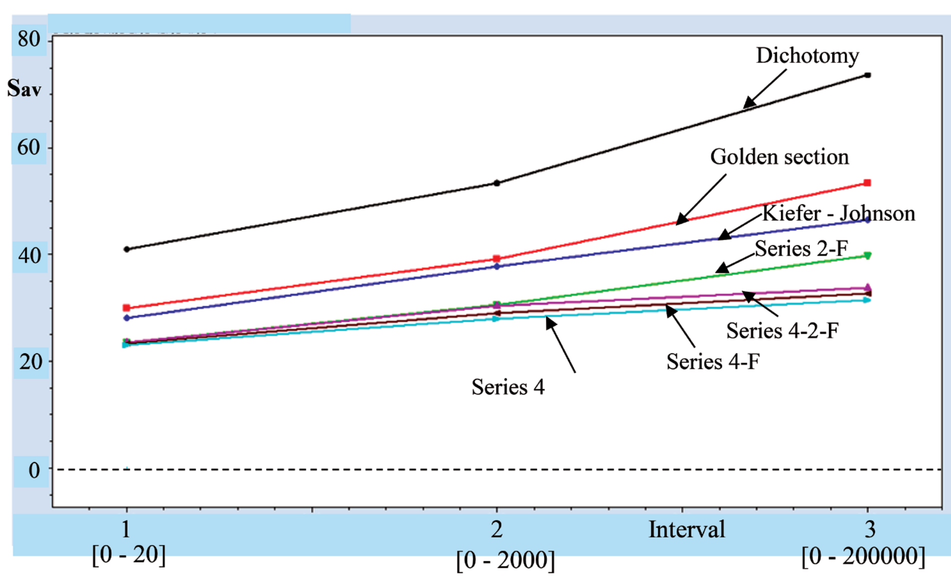

Figure 1. Average number of calculations of the objective function (S_{av}) with a given accuracy Δ for optimum location and different intervals of searching

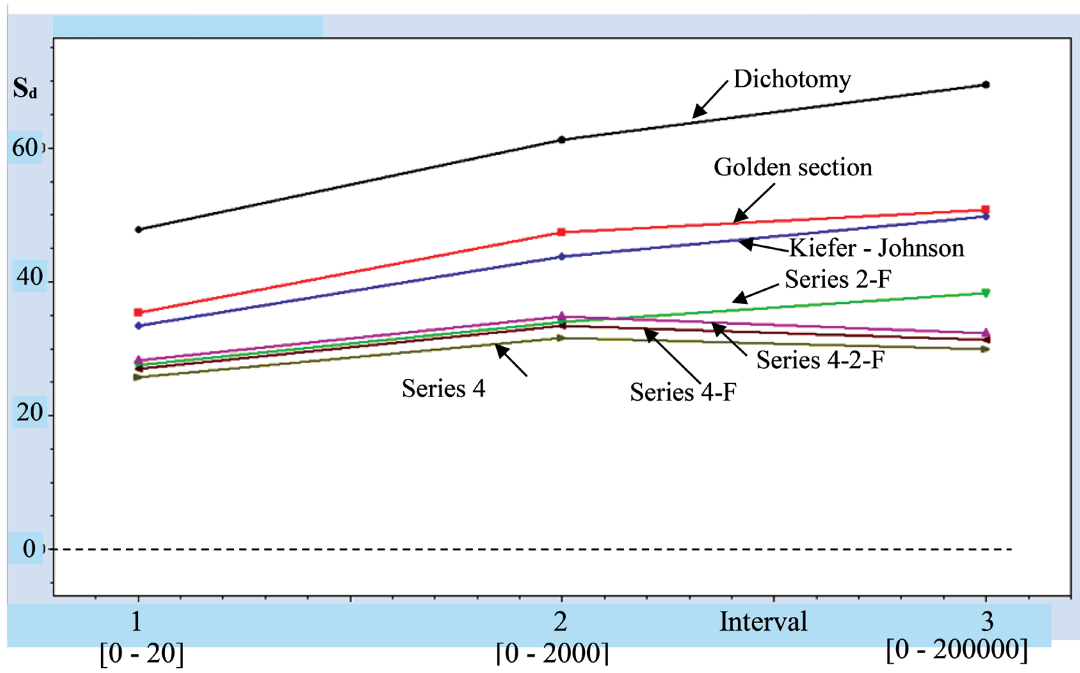

Figure 2. Average number of calculations of the objective function (S_d) at achieved accuracy lg(Δm) and different intervals of searching

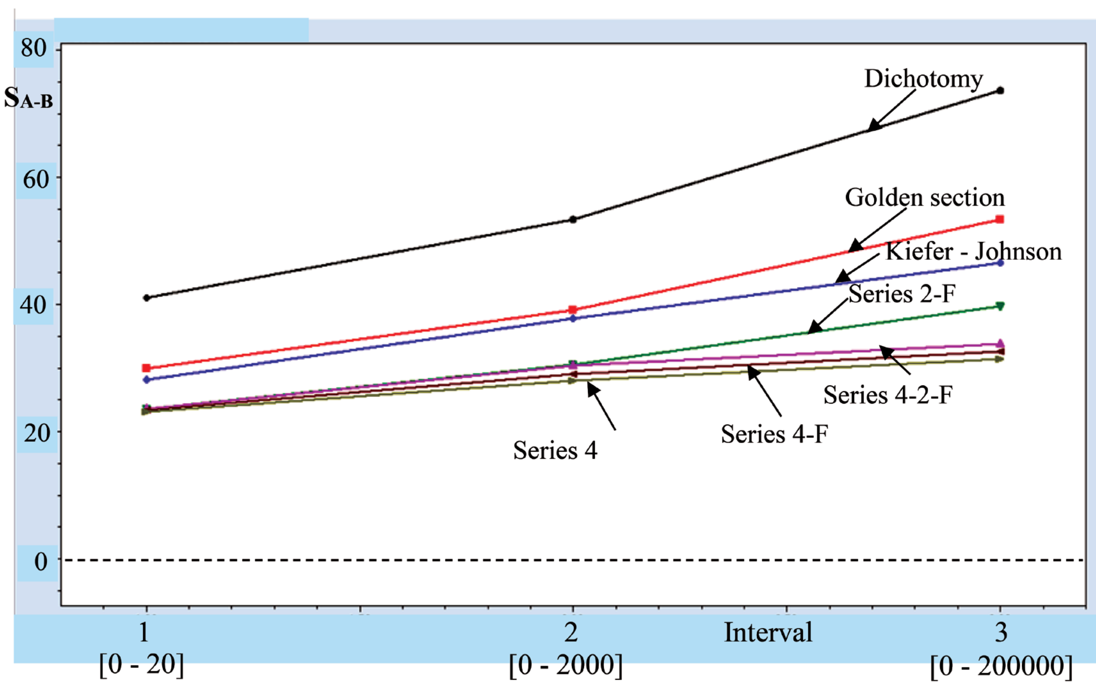

Figure 3. Average number of calculations of the objective function (SA-B) with relative accuracy achieved and different intervals of searching

Conclusion

The study and the obtained results presented on *table 3*, *table 4* and *table 5* and *figure 1*, *figure 2* and *figure 3* show that the proposed new combined functional series (*table 1*), gives a much faster reduction of the uncertainty interval in comparison with the best known in the literature combined series 2-F. The efficiency is much bigger when the uncertainty interval is very large comparing with the efficiency of “dichotomy”, “golden section” and the Kiefer – Johnson method using only Fibonacci numbers.

The new proposed combined series 4-2-F, 4-F and only fourth degree series of “double dichotomy” (series 4), change the trend of the number of objective function evaluations to decrease the average number of calculations when the uncertainty interval for search is too large (*figure 2*).

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